Nonconforming Spline Collocation Methods in Irregular Domains

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This article studies a class of nonconforming spline collocation methods for solving elliptic PDEs in an irregular region with either triangular or quadrilateral partition. In the methods, classical Gaussian points are used as matching points and the special quadrature points in a triangle or quadrilateral element are used as collocation points. The solution and its normal derivative are imposed to be continuous at the marching points. The authors present theoretically the existence and uniqueness of the numerical solution as well as the optimal error estimate in H^1-norm for a spline collocation method with rectangular elements. Numerical results confirm the theoretical analysis and illustrate the high-order accuracy and some superconvergence features of methods. Finally the authors apply the methods for solving two physical problems in compressible flow and linear elasticity, respectively. © 2007 Wiley Periodicals, Inc. Numer Methods Partial Differential Eq 23: 1509–1529, 2007

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I. INTRODUCTION

The spline collocation method is one of the most competitive numerical methods for solving differential equations. This method has evolved as a valuable technique for many physical problems [1–6]. The popularity of such a method is in part due to its ease of implementation and high-order accuracy. One obvious advantage of the spline collocation method over the finite element method is that the calculation of the coefficient matrices is very efficient since no integrals need to be evaluated or approximated. During the past 30 years, considerable research has been devoted to collocation methods and their applications in solving differential equations and partial differential equations in a rectangular region (e.g., see the review paper [7, 8]). The latter is mainly based on the use of the tensor product of one-dimensional approximation.

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The study on efficient spline collocation methods for solving PDEs in a complex region has been an attractive area, but it seems not very successful. A potential problem in a nonrectangular region was first studied in [9] by using orthogonal spline collocation methods. This approach is based on the use of a local curvilinear coordinate system, in which each element is rectangular essentially. Numerical results for certain nonrectangular regions show the high-order accuracy of this method. A global geometrical mapping method was proposed in [10] where nonrectangular regions are transformed to rectangles, and then tensor product Hermite spline collocation methods are applied for the transformed problems in a rectangular region. Methods in terms of coordinate transformations seem not easy for solving problems in complex regions. Besides, the accuracy may be lost because of the approximation of coordinate transformations. In [11], a domain-imbedding alternating direction method was proposed for solving elliptic partial differential equations over bounded two-dimensional irregular regions. The scheme imbeds the irregular region in a rectangle and then uses an alternating direction iteration to solve the resulting system of linear equations. A collocation method with cubic Hermite splines is used for discretization.

For a complex physical region, the most popular practise is to partition it into a triangular or quadrilateral grid, in which small triangles or quadrilaterals with edges oriented along the boundary are used to handle the irregular boundary. Finite element methods (FEM) with such a grid have been well developed and widely applied for solving PDEs and many physical problems. However, the problem for collocation approaches is extremely difficult for such a grid. For a simple linear boundary value problem

$$\begin{aligned} Lu &= f(z), \quad z \in \Omega, \\ u(z) &= 0, \quad z \in \partial\Omega, \end{aligned} \tag{1.1}$$

let

$$u_h(z) = \sum_{i=1}^{N} u_i \phi_i(z)$$

be the numerical solution, where Ω is a bounded domain in R^2 (or R^3) and $\{\phi_i(z)\}$ is the base of a certain spline space satisfying the homogeneous boundary conditions in (1.1). A discrete spline collocation system is given by

$$Lu_h(z_{c,j}) = f(z_{c,j}), \quad j = 1, 2, \ldots, M,$$

where $z_{c,j}$, $j = 1, 2, \ldots, M$, define the collocation points. A major problem is that for most classical spline spaces in FEMs, the collocation systems usually are not unisolvent and, $N \neq M$ in general. Recently, a collocation method based on a nonlocal spline approximation was first introduced by Doedel [12] and in this method, the number of collocation points always equals to the number of unknowns, i.e., $M = N$. The method has been applied for solving some nonlinear Poisson type equations and numerical results with rectangular meshes were presented in [12, 13]. However, no analysis has been obtained.

This article reformulates the spline collocation method for PDEs in an irregular region with either triangular or quadrilateral partition. The method can be viewed as a certain collocation approximation in a class of nonconforming finite element (spline) spaces. The solution in the nonconforming space is discontinuous in general and connected only at those so-called matching points. In this method, the spline collocation system is generated by collocating the partial differential equation at some interior points. In addition, a local elimination is employed to reduce the number of freedoms at each element such that the dimension of the resulting collocation system is the same as the number of matching points. More important is that we prove theoretically the existence and uniqueness of the numerical solution and provide an optimal error estimate in an energy norm for a nonconforming spline collocation method with a rectangular mesh. Numerical results confirm our theoretical analysis and show that the method is efficient for problems in irregular regions. Moreover, this method provides a high-order accuracy when collocation points are chosen to be certain special quadrature points in a triangular or quadrilateral element and matching points are chosen to be the classical Gaussian points. Some interesting superconvergence features have also been observed.

The rest of this article is organized as follows. In section II, we reformulate the nonconforming spline collocation method for the second-order PDEs (1.1). In section III, we reduce a spline collocation system to a equivalent finite difference system in the case of a rectangular mesh on a rectangular domain. We prove the existence and uniqueness of the numerical solution. The optimal error estimate in H^1-norm is obtained in terms of the truncation error and an error in discrete energy norm of the finite difference system. Numerical results with different approximations on both triangular and rectangular partitions are given in section IV. Finally, we apply the nonconforming spline collocation method for certain physical problems, including fluid dynamics and linear elasticity.

II. NONCONFORMING SPLINE COLLOCATION METHODS

In this section, we reformulate the nonlocal spline collocation method in a simpler form. For simplicity, we consider the model in (1.1).

We assume that $\Pi_h = \{e_k\}$ is a regular partition on Ω, consisting of triangles or quadrilaterals, such that $\Omega = \cup e_k$. Two typical elements are shown in Fig. 1. Denote by $z^k_{m,i} (i = 1, 2, \ldots, n_m)$ and $z^k_{c,i} (i = 1, 2, \ldots, n_c)$ the matching points and the collocation points in element e_k, respectively. Let the collocation solution be defined by

$$u_h(z) = \sum_{j=1}^{n_m+n_c} u_{kj}\phi_j(z), \quad z \in e_k, \tag{2.1}$$

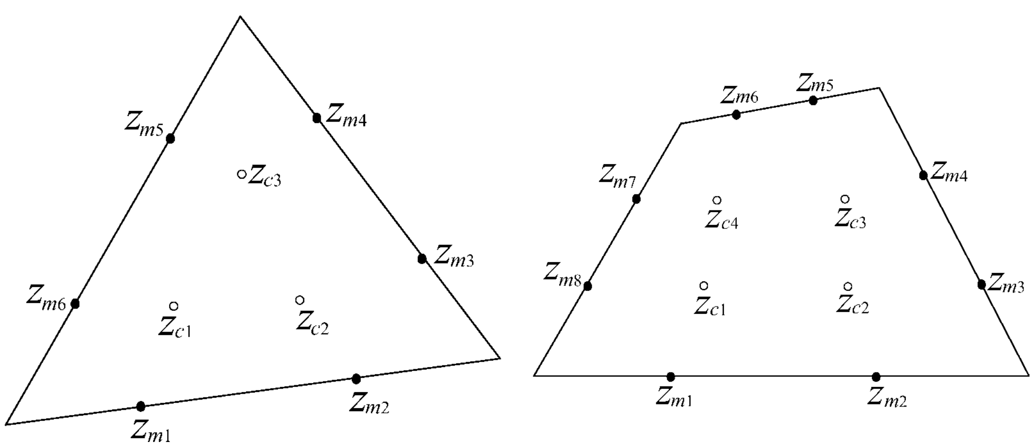

FIG. 1. Two typical nonconforming elements.

where $\phi_j(z)$, $j = 1, 2, \ldots, n_m + n_c$, are basis functions of the polynomial space $P_{n_m+n_c}$. By collocating equation (1.1) at the collocation points, we have a local system

$$L(u_h)\big(z_{c,i}^k\big) = f\big(z_{c,i}^k\big), \quad i = 1, 2, \ldots, n_c \tag{2.2}$$

or in matrix form,

$$C_k u^k = F^k,$$

where $C_k = (c_{ij}^k)$ is an $n_c \times (n_c + n_m)$ matrix and

$$c_{ij}^k = L\phi_j\big(z_{c,i}^k\big), \quad i = 1, 2, \ldots, n_c, \ j = 1, 2, \ldots, n_m + n_c.$$

Let $C_k = (C_{k1}, C_{k2})$ and $u^k = \big(u_1^k, u_2^k\big)^T$, where C_{k1} and C_{k2} are $n_c \times n_m$ and $n_c \times n_c$ matrices, respectively. We obtain

$$C_{k2} u_2^k = F^k - C_{k1} u_1^k,$$

from which

$$u_2^k = -(C_{k2})^{-1} C_{k1} u_1^k + (C_{k2})^{-1} F^k := \bar{C}_k u_1^k + \bar{F}^k$$

or equivalently

$$u_{ki} = \sum_{l=1}^{n_m} \bar{c}_{il}^k u_{kl} + \bar{f}_i^k, \quad i = n_m + 1, n_m + 2, \ldots, n_m + n_c,$$

where $\bar{C}_k = (\bar{c}_{il}^k)$ and $\bar{F}^k = (\bar{f}_i^k)$. The collocation solution can be re-expressed by

$$u_h(z) = \sum_{i=1}^{n_m} u_{ki} \bar{\phi}_i(z) + f^k(z), \tag{2.3}$$

where

$$\bar{\phi}_i(z) = \phi_i(z) + \sum_{j=n_m+1}^{n_m+n_c} \bar{c}_{ji} \phi_j(z), \qquad i = 1, 2, \ldots, n_m,$$

$$f^k(z) = \sum_{j=n_m+1}^{n_m+n_c} \bar{f}_j^k \phi_j(z).$$

By forcing the normal derivative of the collocation solution to be continuous at matching points $z_{m,i}^k$, $i = 1, 2, \ldots, n_m$, we obtain the collocation system

$$\left.\frac{\partial u_h}{\partial n}\right|_{z=z_i^{k+}} + \left.\frac{\partial u_h}{\partial n}\right|_{z=z_i^{k-}} = 0. \tag{2.4}$$

A question immediately arising from the spline collocation method is how to generate basis functions. The approach in [1, 12] is based on the use of the monomial basis functions, $1, x, y, xy, x^2, \ldots$, and the following formula

$$\left.\frac{\partial u_h}{\partial n}\right|_{z=z_{m,i}^k} = \sum_{j=1}^{n_m} \alpha_{ij} u_h\big(z_{m,j}^k\big) + \sum_{j=1}^{n_c} \beta_{ij} f\big(z_{c,j}^k\big),$$

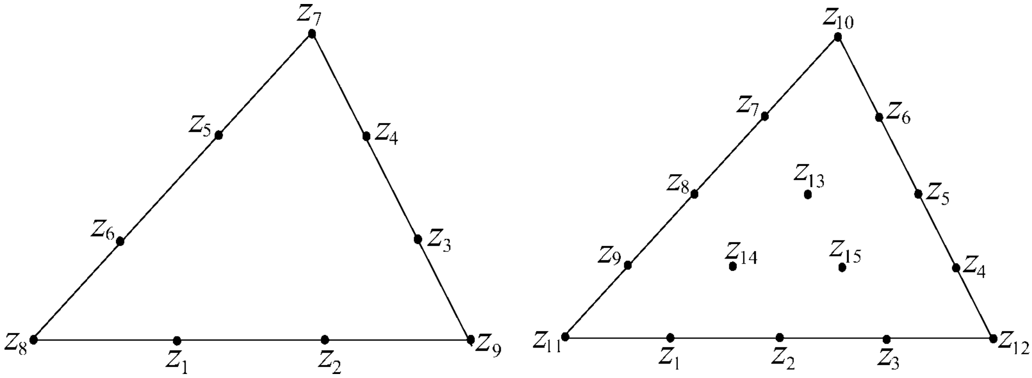

FIG. 2. Interpolation points in triangular elements.

in which one needs to calculate the inverse of an $(n_c + n_m) \times (n_c + n_m)$ matrix at each element to get α_{ij} and β_{ij}. The formulation based on the monomial basis functions may not be efficient for problems in complex domains. A shape function technique in terms of area coordinates or isoparametric representation has been well studied and widely applied in finite element methods to generate basis functions in high-order spline spaces (e.g., see [14]). The formula (2.1) can be considered as an Lagrange interpolation on $\{z_i^k\}_{i=1}^{n_c+n_m}$. The interpolation points may be different from collocation points and matching points. In many high-order approximations, the basis functions $\phi_j(z)$ can be generated in terms of the shape function technique. Two examples are given in Fig. 2. For the 15-node triangular element, one can easily generate 15 quartic shape functions corresponding to the 15 interpolation points z_i with $\phi_i(z_j) = \delta_{ij}$ where z_i, $i = 1, 2, \ldots, 9$, are matching points and six collocation points are chosen on quadrature points of a triangle as shown in Table I. Here collocation points are not the interpolation points. After the local elimination process (2.3), six unknowns corresponding three interior points and three vertices are eliminated in terms of the six collocation equations in (2.2). In this approach, the number of freedom at each element reduces from $n_m + n_c$ to n_m, the same as the number of matching points on the edges of an element. only those u_{ki} corresponding to matching points are involved in the resulting collocation system. It has been noted that this local elimination technique has been widely used in spline collocation methods, such as by Ascher et al. [15] to reduce the size of collocation systems in system of ODEs, by Mahmood and Osborne [16] to construct a compact formula and by the first author [2, 17] to develop a class of block iterative algorithms for solving the spline collocation systems.

The spline collocation linear system (2.4) seems much different from those in finite difference and finite element methods. The coefficient matrix in this system is sparse and only the unknowns in two adjacent elements are involved in each equation. Classical orthogonal spline collocation methods strongly depend upon the choice of collocation points. High-order accuracy can be obtained only when corresponding Gaussian points are employed. Usually there are two different ways to find (Gaussian) quadrature points. One is based on zeros of a certain orthogonal polynomial and other one is to generate a quadrature rule which is exact for polynomials of degree as high as possible. These two approaches are equivalent in a one-dimensional space or a rectangular region. Quadrature rules in a triangle were studied by several authors [18, 19], either by using a transformation from a triangle to a rectangle or by generating a quadrature rule being exact for any polynomials of degree as high as possible. We list these special quadrature points and corresponding weights in Table I. In the nonconforming spline collocation method,

TABLE I. The quadrature points and weights in an area coordinate system.

Number of points	Weight	Area coordinates (ξ, η, ζ)
1	1	(1/3,1/3,1/3)
3	1/3	(2/3,1/6,1/6)
	1/3	(2/3,2/3,2/3)
	1/3	(2/3,1/6,1/6)
4	−27/48	(1/3,1/3,1/3)
	25/48	(0.6,0.2,0.2)
	25/48	(0.2,0.6,0.2)
	25/48	(0.2,0.2,0.6)
6	ω_1, ω_1	$(\alpha_1, \alpha_2, \alpha_2)$, $(\alpha_2, \alpha_1, \alpha_2)$
	ω_1, ω_2	$(\alpha_2, \alpha_2, \alpha_1)$, $(\alpha_3, \alpha_4, \alpha_4)$
	ω_2, ω_2	$(\alpha_4, \alpha_3, \alpha_4)$, $(\alpha_4, \alpha_4, \alpha_3)$
	$\alpha_1 = 0.816847572980459$, $\alpha_2 = 0.091576213509777$, $\alpha_3 = 0.108103018168070$, $\alpha_4 = 0.445948490915965$, $\omega_1 = 0.10995174365$, $\omega_2 = 0.22338158967$	

we shall choose classical Gaussian points as matching points and these special points in Table I as collocation points in all computations.

III. ANALYSIS FOR RECTANGULAR PARTITION

In this section, we present the error analysis of a special nonconforming spline collocation method for the model problem

$$\begin{aligned} -\Delta u &= f(x, y), \quad (x, y) \in \Omega, \\ u(x, y) &= 0, \quad (x, y) \in \partial\Omega, \end{aligned} \tag{3.1}$$

in a rectangular domain with rectangular elements. For simplicity, we assume $\Omega = [0, 1] \times [0, 1]$. Let $\Pi_h = \{x_i, y_j\}_{i,j=0}^{N}$ be a uniform partition on Ω with the mesh size $h = 1/N$. Define by

$$x = x_i + h\sigma_1, \qquad y = y_j + h\sigma_2$$

the local coordinate system (σ_1, σ_2) on each element with $\sigma_1, \sigma_2 \in [0, 1]$.

A typical element and the connection of four elements are depicted in Fig. 3. There are two matching points on each edge and four collocation points in each element and consequently, $N_m = 4N(N-1)$ interior matching points and $N_c = 4N^2$ collocation points in Ω. Denote by $\Omega_h = \{z_i\}$ the set of matching points in Ω and $\partial\Omega_h \subset \Omega_h$ the set of matching points in the elements adjacent to the boundary $\partial\Omega$. Let

$$\tau_1 = \frac{3-\sqrt{3}}{6}, \qquad \tau_2 = \frac{3+\sqrt{3}}{6}$$

be two Gaussian points in $(0, 1)$. In each element, the eight marching are defined by $(\tau_i, 0)$, $(\tau_i, 1)$, $(0, \tau_i)$, $(1, \tau_i)$, $i = 1, 2$, and the four collocation points are given by (τ_i, τ_j), $i, j = 1, 2$. Now we seek a nonconforming spline collocation solution u_h, such that (i) in each element $e_k \in \Pi_h$,

$$\begin{aligned} &u_h|_{e_k} \in P_{8+4} := \text{span}\left\{1, \sigma_1, \sigma_2, \sigma_1^2, \sigma_1\sigma_2, \sigma_2^2, \sigma_1^3, \sigma_1^2\sigma_2, \sigma_1\sigma_2^2, \sigma_2^3, \sigma_1^3\sigma_2, \sigma_1\sigma_2^3\right\}, \\ &-\Delta u_h\left(z_{c_i}^k\right) = f\left(z_{c_i}^k\right), i = 1, 2, 3, 4; \end{aligned} \tag{3.2}$$

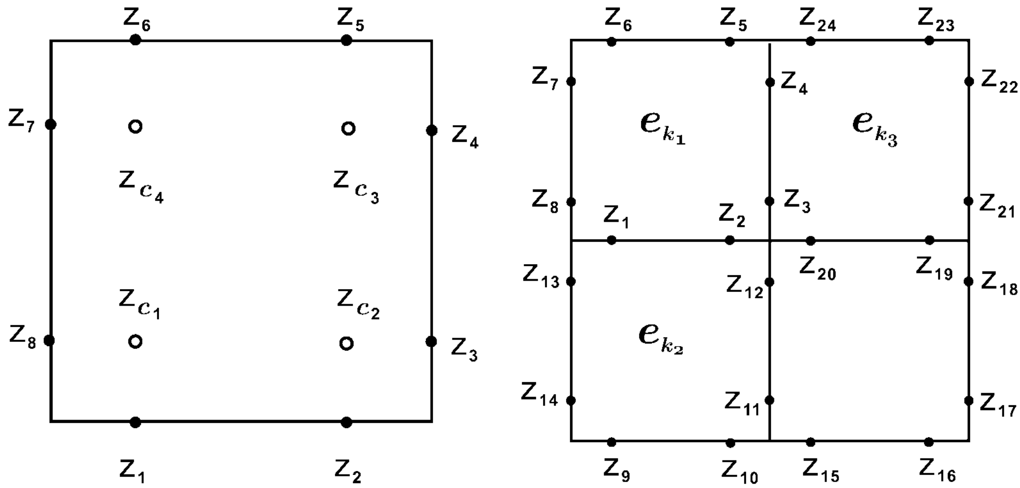

FIG. 3. Rectangular elements and their connection.

(ii) u_h and its normal derivative are continuous at the matching points in Ω_h. Besides, the homogeneous boundary condition is imposed so that u_h vanishes at the matching points on $\partial\Omega$.

Let $u_i = u_h(z_i)$. In an element e_k, by (2.3) the normal derivative on the matching point z_i^k is given by

$$\left.\frac{\partial u_h}{\partial n}\right|_{z=z_i^k} = \sum_{j=1}^{8} u_{kj} \left.\frac{\partial \bar{\phi}_i(z)}{\partial n}\right|_{z=z_i^k} := (Au^k)_i + (BF^k)_i \tag{3.3}$$

where $(v)_i$ denotes the ith component of the vector v, A is an 8×8 matrix, B is an 8×4 matrix, and $F^k = (f(z_{c_1}^k), f(z_{c_2}^k), f(z_{c_3}^k), f(z_{c_4}^k))^T$. By *Maple* software, we obtain

$$A = -\frac{1}{4h}\begin{bmatrix} -16 & 3 & 0 & a^- & 0 & 1 & 0 & a^+ \\ 3 & -16 & a^+ & 0 & 1 & 0 & a^- & 0 \\ 0 & a^+ & -16 & 3 & 0 & a^- & 0 & 1 \\ a^- & 0 & 3 & -16 & a^+ & 0 & 1 & 0 \\ 0 & 1 & 0 & a^+ & -16 & 3 & 0 & a^- \\ 1 & 0 & a^- & 0 & 3 & -16 & a^+ & 0 \\ 0 & a^- & 0 & 1 & 0 & a^+ & -16 & 3 \\ a^+ & 0 & 1 & 0 & a^- & 0 & 3 & -16 \end{bmatrix} \tag{3.4}$$

and

$$B = -\frac{1}{96h}\begin{bmatrix} b_1^+ & b_2^+ & b_2^- & b_1^- \\ b_2^+ & b_1^+ & b_1^- & b_2^- \\ b_1^- & b_1^+ & b_2^+ & b_2^- \\ b_2^- & b_2^+ & b_1^+ & b_1^- \\ b_2^- & b_1^- & b_1^+ & b_2^+ \\ b_1^- & b_2^- & b_2^+ & b_1^+ \\ b_2^+ & b_2^- & b_1^- & b_1^+ \\ b_1^+ & b_1^- & b_2^- & b_2^+ \end{bmatrix}$$

where $a^{\pm} = 6 \pm 3\sqrt{3}$, $b_1^{\pm} = 9 \pm 5\sqrt{3}$, and $b_2^{\pm} = 3 \pm \sqrt{3}$. A four-element diagram is shown in Fig. 3. The matching point z_1 locates at the interface of elements e_{k_1} and e_{k_2}. Then the continuity condition of the normal derivative at z_1 results in

$$(Au^{k_1})_1 + (BF^{k_1})_1 + (Au^{k_2})_6 + (BF^{k_2})_6 = 0$$

or equivalently,

$$\begin{aligned}
&\frac{1}{4h}\left(16u_1 - 3u_2 - (6-3\sqrt{3})u_4 - u_6 - (6+3\sqrt{3})u_8\right) \\
&\quad - \frac{h}{96}\left((9+5\sqrt{3}) f_{k_1,1} + (3+\sqrt{3}) f_{k_1,2} + (3-\sqrt{3}) f_{k_1,3} + (9-5\sqrt{3}) f_{k_1,4}\right) \\
&+ \frac{1}{4h}\left(-u_9 - (6-3\sqrt{3})u_{11} - 3u_2 + 16u_1 - (6+3\sqrt{3})u_{13}\right) \\
&\quad - \frac{h}{96}\left((9-5\sqrt{3}) f_{k_2,1} + (3-\sqrt{3}) f_{k_2,2} + (3+\sqrt{3}) f_{k_2,3} + (9+5\sqrt{3}) f_{k_2,4}\right) = 0,
\end{aligned}$$

where $f_{k,i} = f(z_{c_i}^k)$, $k = k_1, k_2$. The equations at $z_2, z_3, z_4, \ldots$ can be obtained in a similar way. Thus the collocation system reduces to a finite difference system

$$-\Delta_h u_h(z_i) = \widetilde{f}_i, \quad z_i \in \Omega_h, \tag{3.5}$$

where

$$\begin{aligned}
-\Delta_h u_h(z_1) &:= 32u_1 - 6u_2 - (6-3\sqrt{3})(u_4+u_{11}) - u_6 - u_9 - (6+3\sqrt{3})(u_8+u_{13}) \\
&= \frac{h^2}{24}((9+5\sqrt{3})(f_{k_1,1}+f_{k_2,4}) + (9-5\sqrt{3})(f_{k_1,4}+f_{k_2,1}) + (3+\sqrt{3})(f_{k_1,2}+f_{k_2,3}) \\
&+ (3-\sqrt{3})(f_{k_1,3}+f_{k_2,2})) := \widetilde{f}_1, \\
-\Delta_h u_h(z_2) &:= 32u_2 - 6u_1 - (6-3\sqrt{3})(u_7+u_{14}) - u_5 - u_{10} - (6+3\sqrt{3})(u_3+u_{12}) \\
&= \frac{h^2}{24}((9+5\sqrt{3})(f_{k_1,2}+f_{k_2,3}) + (9-5\sqrt{3})(f_{k_1,3}+f_{k_2,2}) + (3+\sqrt{3})(f_{k_1,1}+f_{k_2,4}) \\
&+ (3-\sqrt{3})(f_{k_1,4}+f_{k_2,1})) := \widetilde{f}_2, \\
-\Delta_h u_h(z_3) &:= 32u_3 - 6u_4 - (6-3\sqrt{3})(u_6+u_{23}) - u_8 - u_{21} - (6+3\sqrt{3})(u_2+u_{20}) \\
&= \frac{h^2}{24}((9+5\sqrt{3})(f_{k_1,2}+f_{k_3,1}) + (9-5\sqrt{3})(f_{k_1,1}+f_{k_3,2}) + (3+\sqrt{3})(f_{k_1,3}+f_{k_3,4}) \\
&+ (3-\sqrt{3})(f_{k_1,4}+f_{k_3,3})) := \widetilde{f}_3, \\
-\Delta_h u_h(z_4) &:= 32u_4 - 6u_3 - (6-3\sqrt{3})(u_1+u_{19}) - u_7 - u_{22} - (6+3\sqrt{3})(u_5+u_{24}) \\
&= \frac{h^2}{24}((9+5\sqrt{3})(f_{k_1,3}+f_{k_3,4}) + (9-5\sqrt{3})(f_{k_1,4}+f_{k_3,3}) + (3+\sqrt{3})(f_{k_1,2}+f_{k_3,1}) \\
&+ (3-\sqrt{3})(f_{k_1,1}+f_{k_3,2})) := \widetilde{f}_4
\end{aligned} \tag{3.6}$$

and equations on the other matching points in Ω_h have the same structure.

Let $A_c = (a_{ij}^c)$ be the coefficient matrix of the system (3.5). We see that A_c is an $N_m \times N_m$ diagonally dominant and irreducible M-matrix. The existence of solution of the collocation system follows immediately.

Theorem 3.1. *The collocation system (3.5) has a unique solution. Let $u(z)$ denote the exact solution of* (3.1) *and $e(z_i) = u(z_i) - u_i$ be the nodal error and let C denote a generic positive constant independent of h with possibly different values in different places. In terms of Maple software again, we obtain the truncation error*

$$\begin{aligned} -\Delta_h e(z_1) &= -\Psi(z_0)h^4 + O(h^5), \\ -\Delta_h e(z_2) &= -\Psi(z_0)h^4 + O(h^5), \\ -\Delta_h e(z_3) &= \Psi(z_0)h^4 + O(h^5), \\ -\Delta_h e(z_4) &= \Psi(z_0)h^4 + O(h^5), \end{aligned} \tag{3.7}$$

where

$$z_0 = \frac{z_1 + z_{19}}{2}, \quad \Psi = \frac{1}{36}(f_{xx} - f_{yy}). \tag{3.8}$$

Lemma 3.2. *Let $E = (e(z_1), e(z_2), \dots e(z_{N_m}))^T$ and assume that $u \in C^5(\bar{\Omega})$. Then*

$$\sqrt{E^T A_c E} \leq Ch^3, \tag{3.9}$$

where $A_c = (a_{ij}^c)_{N_m \times N_m}$ is the coefficient matrix of the system (3.5).

Proof. Since the matrix A_c is symmetric, we have the classical formulation

$$E^T A_c E = \sum_{i=1}^{N_m} (e(z_i))^2 \left(\sum_{j=1}^{N_m} a_{ij}^c \right) + \sum_{j>i} \left(-a_{ij}^c \right) (e(z_i) - e(z_j))^2 \tag{3.10}$$

and moreover, by noting

$$\begin{aligned} & a_{ii}^c = 32, \ \ a_{ij}^c \leq 0, \ \ 1 \leq i, j \leq N_m, \ \ i \neq j, \\ & \sum_{j=1}^{N_m} a_{ij}^c \geq 1, \quad z_i \in \partial\Omega_h \quad \text{and} \quad \sum_{j=1}^{N_m} a_{ij}^c = 0, \quad z_i \in \Omega_h \backslash \partial\Omega_h, \end{aligned} \tag{3.11}$$

we get

$$E^T A_c E \geq \sum_{z_i \in \partial\Omega_h} (e(z_i))^2 + \sum_{j>i} \left(-a_{ij}^c \right) (e(z_i) - e(z_j))^2. \tag{3.12}$$

On the other hand, by noting (3.7), we can rewrite the quadratic form by

$$E^T A_c E = \sum_{i=1}^{N_m} (-\Delta_h e(z_i)) e(z_i) := h^4 \Sigma_1 + h^5 \Sigma_2. \tag{3.13}$$

Note that each of Σ_l, $l = 1, 2$, contains N_m terms and each term corresponds to a matching point.

For Σ_1, we combine all the matching points in Ω_h into a set of specific pairs, (z_1, z_4), (z_2, z_3), (z_{11}, z_{19}), (z_{12}, z_{20}), ... For each pair, we get an estimate for the corresponding terms in Σ_1 such as

$$\Psi(z_0)h^4(e(z_1) - e(z_4)) \leq \frac{1}{2\varepsilon}\Psi^2(z_0)h^8 + \frac{\varepsilon}{2}(e(z_1) - e(z_4))^2$$
$$\Psi(z_0)h^4(e(z_2) - e(z_3)) \leq \frac{1}{2\varepsilon}\Psi^2(z_0)h^8 + \frac{\varepsilon}{2}(e(z_2) - e(z_3))^2$$
$$\Psi(z_0)h^4(e(z_{11}) - e(z_{19})) \leq \frac{1}{2\varepsilon}\Psi^2(z_0)h^8 + \frac{\varepsilon}{2}(e(z_{11}) - e(z_{19}))^2$$
$$\Psi(z_0)h^4(e(z_{12}) - e(z_{20})) \leq \frac{1}{2\varepsilon}\Psi^2(z_0)h^8 + \frac{\varepsilon}{2}(e(z_{12}) - e(z_{20}))^2,$$

where ε is a small positive constant to be determined later. By summarizing the above inequalities, we get

$$h^4|\Sigma_1| \leq C\varepsilon^{-1}h^6 + C\varepsilon\sum_{j>i}\big(-a^c_{ij}\big)(e(z_i) - e(z_j))^2 \leq C\big(\varepsilon^{-1}h^6 + \varepsilon E^T A_c E\big), \qquad (3.14)$$

where (3.12) has been used.

As for Σ_2, by noting the structure of A_c, for any given $z_j \in \Omega_h$, there exists a chain $(z_j, z_{j_1}, \ldots, z_{j_{l_j}})$ with $l_j \leq N$ such that $a^c_{j_{i-1},j_i} = -1$ and $z_{j_{l_j}} \in \partial\Omega_h$. Then we have

$$|e(z_j)| \leq \sum_{i=1}^{l_j}|e(z_{j_i}) - e(z_{j_{i-1}})| + |e\big(z_{j_{l_j}}\big)|$$
$$= \sum_{i=1}^{l_j}\sqrt{-a^c_{j_{i-1},j_i}}|e(z_{j_i}) - e(z_{j_{i-1}})| + |e\big(z_{j_{l_j}}\big)|$$

and therefore,

$$\sum_{j=1}^{N_m}|e(z_j)| \leq \sum_{j=1}^{N_m}\left[\sum_{i=1}^{l_j}\sqrt{-a^c_{j_{i-1},j_i}}|e(z_{j_i}) - e(z_{j_{i-1}})| + |e\big(z_{j_{l_j}}\big)|\right]$$
$$\leq CN\left[\sum_{j>i,\ a^c_{ij}=-1}\sqrt{-a^c_{ij}}|e(z_i) - e(z_j)| + \sum_{z_i\in\partial\Omega_h}|e(z_i)|\right]$$
$$\leq CN^2\sqrt{E^T A_c E}, \qquad (3.15)$$

where we have used (3.12) again. Hence,

$$h^5|\Sigma_2| \leq Ch^5\sum_{i=1}^{N_m}|e(z_i)| \leq Ch^3\sqrt{E^T A_c E} \leq C(\varepsilon^{-1}h^6 + \varepsilon E^T A_c E). \qquad (3.16)$$

Substituting (3.14) and (3.16) into (3.13) gives

$$E^T A_c E \leq C\big(\varepsilon^{-1}h^6 + \varepsilon E^T A_c E\big),$$

which leads to (3.9) when ε is selected to be small enough, say, less than $1/(2C)$. ■

To derive a global error estimate, we introduce some notations. For a function u on Ω, we define two semi-norms, the energy norm

$$\|u\|_1 := \left(\sum_{e_k\in\Pi_h} \|u\|_{1,e_k}^2\right)^{1/2} = \left(\sum_{e_k\in\Pi_h} \int_{e_k} |\nabla u|^2 dxdy\right)^{1/2}$$

and the corresponding discrete energy norm

$$\|u\|_{1,h} := \left(\sum_{e_k\in\Pi_h} \|u\|_{1,e_k,h}^2\right)^{1/2} = \left(\frac{h}{2}\sum_{e_k\in\Pi_h} (u^k)^T A u^k\right)^{1/2},$$

where $u^k = (u(z_1^k), u(z_2^k), \dots, u(z_8^k))^T$ is the function vector at the eight matching points and A is defined in (3.4). The equivalence between these two semi-norms are given in the following lemma.

Lemma 3.3. *Let u be a function defined in Ω, $u|_{e_k} \in P_{8+4}$ and $\Delta u(z_{c_i}^k) = 0$, $i = 1, 2, 3, 4$. Then $u(z)$ is harmonic in each element and*

$$\frac{1}{\sqrt{2}}\|u\|_1 \le \|u\|_{1,h} \le \sqrt{2}\|u\|_1. \tag{3.17}$$

Proof. For any $u \in P_{8+4}$, we have $\Delta u \in \text{span}\{1, \sigma_1, \sigma_2, \sigma_1\sigma_2\}$. By noting that Δu vanishes at the four collocation points $(\tau_i, \tau_j)(i, j = 1, 2)$ in each element, we get $\Delta u = 0$. Moreover, in the local coordinate system,

$$u|_{e_k} = \frac{1}{8(\tau_2 - \tau_1)}\boldsymbol{\phi} S^T u^k, \tag{3.18}$$

where

$$\boldsymbol{\phi} = \left(1, \sigma_1, \sigma_2, \sigma_1^2 - \sigma_2^2, \sigma_1\sigma_2, \sigma_1^3 - 3\sigma_1\sigma_2^2, \sigma_2^3 - 3\sigma_1^2\sigma_2, \sigma_1^3\sigma_2 - \sigma_1\sigma_2^3\right),$$

$$S = \begin{bmatrix} 3\tau_2+1 & 30\tau_2-11 & -20\tau_2-7 & -30\tau_2-12 & 40-36\tau_2 & 18 & -s_2 & -24 \\ -3\tau_1-1 & 11-30\tau_1 & 20\tau_1+7 & 30\tau_1+12 & 36\tau_1-40 & -18 & s_1 & 24 \\ \tau_2 & 1-4\tau_2 & -6\tau_2-1 & -6\tau_2-6 & 36\tau_2-4 & s_2 & -6 & -24 \\ -\tau_1 & 4\tau_1-1 & 6\tau_1+1 & 6\tau_1+6 & 4-36\tau_1 & -s_1 & 6 & 24 \\ -\tau_1 & 6\tau_1+1 & 4\tau_1-1 & -6\tau_1-6 & 4-36\tau_1 & 6 & -s_1 & -24 \\ \tau_2 & -6\tau_2-1 & 1-4\tau_2 & 6\tau_2+6 & 36\tau_2-4 & -6 & s_2 & 24 \\ -3\tau_1-1 & 20\tau_1+7 & 11-30\tau_1 & -30\tau_1-12 & 36\tau_1-40 & s_1 & -18 & -24 \\ 3\tau_2+1 & -20\tau_2-7 & 30\tau_2-11 & 30\tau_2+12 & 40-36\tau_2 & -s_2 & 18 & 24 \end{bmatrix}$$

and $s_l = 12\tau_l + 6$, $l = 1, 2$. By Green's theorem,

$$\|u\|_{1,e_k}^2 = \int_{\partial e_k} u\frac{\partial u}{\partial n}\, ds = (u^k)^T T u^k, \tag{3.19}$$

where

$$T = \frac{3}{64} S\left(\int_{\partial e_k} \boldsymbol{\phi}^T \frac{\partial \boldsymbol{\phi}}{\partial n}\, ds\right) S^T.$$

By straightforward calculation with *Maple* software, we have

$$T = \frac{1}{1120}\begin{bmatrix} 2046 & -205 & -236 & t_1 & -278 & 117 & -236 & t_2 \\ -205 & 2046 & t_2 & -236 & 117 & -278 & t_1 & -236 \\ -236 & t_2 & 2046 & -205 & -236 & t_1 & -278 & 117 \\ t_1 & -236 & -205 & 2046 & t_2 & -236 & 117 & -278 \\ -278 & 117 & -236 & t_2 & 2046 & -205 & -236 & t_1 \\ 117 & -278 & t_1 & -236 & -205 & 2046 & t_2 & -236 \\ -236 & t_1 & -278 & 117 & -236 & t_2 & 2046 & -205 \\ t_2 & -236 & 117 & -278 & t_1 & -236 & -205 & 2046 \end{bmatrix}$$

where

$$t_1 = -604 + 441\sqrt{3}, \quad t_2 = -604 - 441\sqrt{3}.$$

The matrix $\frac{h}{2}A - \frac{1}{2}T$ is symmetric and nonnegative definite since its eigenvalues are

$$0, \frac{1}{2}, \frac{3}{2}, \frac{199}{70}, \left(\frac{77}{80} \pm \frac{\sqrt{2509}}{80}\right)_2.$$

Consequently, we have

$$\frac{h}{2}(u_k)^T A u_k \geq \frac{1}{2}(u_k)^T T u_k$$

i.e.,

$$\|u\|^2_{1,e_k,h} \geq \frac{1}{2}\|u\|^2_{1,e_k}.$$

Similarly, we can show that $2T - \frac{h}{2}A$ is symmetric and nonnegative definite and therefore,

$$\|u\|^2_{1,e_k,h} \leq 2\|u\|^2_{1,e_k}.$$

Summarizing the above two inequalities over all the elements in Π_h leads to (3.17), which completes the proof. ■

Let u_I be an interpolant to u such that, in each element e_k in Π_h,

$$u_I|_{e_k} \in P_{8+4}, \quad u_I(z_i) = u(z_i), \quad 1 \leq i \leq 8,$$
$$\Delta u_I\left(z^k_{c_j}\right) = \Delta u\left(z^k_{c_j}\right), \quad 1 \leq j \leq 4. \tag{3.20}$$

By (3.18), we find that u_I is uniquely determined by the condition (3.20) and therefore, $u_I = u$ when u is a polynomial of degree less than 4. The error estimate of the interpolant u_I follows immediately from the classical interpolation theory.

Lemma 3.4. *Assume that $u \in H^4(\Omega)$ and u_I is the interpolant of u defined by* (3.20). *Then*

$$\|u - u_I\|_1 \leq Ch^3. \tag{3.21}$$

Our main theorem on the error estimate of the nonconforming spline collocation method is given below.

Theorem 3.5. *Assume that* $u(z) \in C^5(\bar{\Omega})$ *is the solution of* (3.1) *and* $u_h(z)$ *is the nonconforming spline collocation solution. Then*

$$\|u - u_h\|_1 \leq Ch^3. \tag{3.22}$$

Proof. By Lemma 3.3,

$$\|u_h - u_I\|_1 \leq \sqrt{2}\|u_h - u_I\|_{1,h} = \sqrt{2}\left(\sum_{e_k \in \Pi_h} \|u_h - u_I\|^2_{1,e_k,h}\right)^{1/2}.$$

Let $E = (e(z_1), e(z_2), \ldots, e(z_{N_m}))^T$ denote the error vector. Note that $(u_I - u_h)(z_i) = e(z_i)$, $z_i \in \Omega_h$.

From the definition of Δ_h, we have

$$\begin{aligned} E^T A_c E &= \sum_{z_i \in \Omega_h} (-\Delta_h e(z_i)) \cdot e(z_i) \\ &= \sum_{z_i \in \Omega_h} 4h\big(\big(AE_{e^+_{z_i}}\big)_{z_i} + \big(AE_{e^-_{z_i}}\big)_{z_i}\big) \cdot e(z_i) \\ &= \sum_{e_k \in \Pi_h} 4hE_k^T AE_k = 8 \sum_{e_k \in \Pi_h} \|u_h - u_I\|^2_{1,e_k,h}, \end{aligned} \tag{3.23}$$

where A_c is the coefficient matrix of the system (3.5), $E_k = (e(z_1^k), e(z_2^k), \ldots, e(z_8^k))^T$ denotes the error vector at the eight matching points on e_k, and $e^+_{z_i}$ and $e^-_{z_i}$ are the two elements that share the same matching point z_i. By Lemma 3.2,

$$\|u_h - u_I\|_1 \leq Ch^3.$$

Hence (3.22) is obtained from the above inequality and Lemma 3.4. ■

IV. NUMERICAL EXPERIMENTS

In this section, computational results are reported for four examples to show the accuracy and applicability of the nonconforming spline collocation methods. Throughout, the computation is performed on a Blade 1000 Sun-workstation in double precision.

Example 4.1. First we consider a simple example: the Poisson's equation (3.1) in a unit square domain. The right hand side function $f(x, y)$ and the boundary condition are calculated from the given exact solutions. We use a uniform square partition and the nonconforming spline collocation method described in Section 3. Several error measures are defined by

$$\|e\|_{L^2} = \left(\int_\Omega (u - u_h)^2\, dxdy\right)^{1/2},$$

$$\|e\|_{L^\infty} = \max_{z\in\Omega}\{|u(z) - u_h(z)|\},$$

$$\|e\|_\infty = \max_{z_i\in\Omega_h}\{|u(z_i) - u_h(z_i)|\},$$

$$\|e\|_{H^1} = \left(\int_\Omega (u - u_h)^2\, dxdy + \int_\Omega |\nabla u - \nabla u_h|^2 dxdy\right)^{1/2}.$$

TABLE II. The accuracy of the method with square elements (Example 4.1, Gaussian type).

error	$h = 1/4$	$h = 1/8$	$h = 1/16$	order (h^α)
$u = \sin(\pi x)\sin(\pi y)$				
$\|e\|_{L^2}$	4.020D-4	2.507D-5	1.566D-6	4.00
$\|e\|_{H^1}$	1.291D-2	1.634D-3	2.049D-4	3.00
$\|e\|_{L^\infty}$	1.672D-3	1.168D-4	7.501D-6	3.96
$\|e\|_\infty$	2.859D-4	1.687D-5	1.038D-6	4.02
$u = \sin(x + 2y)$				
$\|e\|_{L^2}$	6.436D-5	3.991D-6	2.489D-7	4.00
$\|e\|_{H^1}$	1.429D-3	1.790D-4	2.239D-5	3.00
$\|e\|_{L^\infty}$	1.534D-4	9.789D-6	6.291D-7	3.96
$\|e\|_\infty$	5.895D-5	4.009D-6	2.586D-7	3.95
$u = 10e^x\cos(y)$				
$\|e\|_{L^2}$	1.480D-4	9.266D-6	5.794D-7	4.00
$\|e\|_{H^1}$	4.103D-3	5.134D-4	6.419D-5	3.00
$\|e\|_{L^\infty}$	7.886D-4	5.162D-5	3.296D-6	3.97
$\|e\|_\infty$	8.098D-6	2.731D-7	8.950D-9	4.93

We present numerical results in Table II with three different exact solutions, where we have listed errors and orders of accuracy in the norms defined above. It is clear to see that the accuracy of the method in H^1 norm (equivalently in the energy norm defined in section 3) for all three cases is $O(h^3)$, which confirms our theoretical analysis in the above section. Numerical results also show that the accuracy of the method are $O(h^4)$ in $\|\cdot\|_{L^2}$ and $\|\cdot\|_{L^\infty}$ norms for all three cases and $O(h^4)$ in $\|\cdot\|_\infty$ norm for the first two cases, although we cannot provide the theoretical analysis. More interesting is the superconvergence phenomenon that the accuracy of the method on matching points for the third case is $O(h^5)$, which is better than those for the first two cases. The main reason is that the solution in this case is harmonic, while orthogonal cubic spline collocation method does not produce a higher order accuracy for harmonic functions.

It has been noted that numerical accuracy of spline collocation methods depends upon the choice of collocation points. Collocation methods on Gaussian points give better accuracy in general. We present numerical results of the method with non-Gaussian collocation points in Table III,

TABLE III. The accuracy of the method with square elements (Example 4.1, non-Gaussian type).

error	$h = 1/4$	$h = 1/8$	$h = 1/16$	order (h^α)
$u = \sin(\pi x)\sin(\pi y)$				
$\|e\|_{L^2}$	1.097D-2	2.806D-3	7.054D-4	1.99
$\|e\|_{H^1}$	5.402D-2	1.304D-2	3.229D-3	2.01
$\|e\|_{L^\infty}$	2.162D-2	5.589D-3	1.410D-3	1.99
$\|e\|_\infty$	2.145D-2	5.582D-3	1.409D-3	1.99
$u = \sin(x + 2y)$				
$\|e\|_{L^2}$	1.368D-3	3.518D-4	8.859D-5	1.99
$\|e\|_{H^1}$	6.566D-3	1.641D-3	4.104D-4	2.00
$\|e\|_{L^\infty}$	2.551D-3	6.480D-4	1.626D-4	1.99
$\|e\|_\infty$	2.540D-3	6.473D-4	1.626D-4	1.99
$u = 10e^x\cos(y)$				
$\|e\|_{L^2}$	2.739D-3	6.876D-4	1.721D-4	2.00
$\|e\|_{H^1}$	1.287D-2	3.229D-3	8.087D-4	2.00
$\|e\|_{L^\infty}$	5.819D-3	1.295D-3	3.159D-4	2.04
$\|e\|_\infty$	4.854D-3	1.237D-3	3.123D-4	1.99

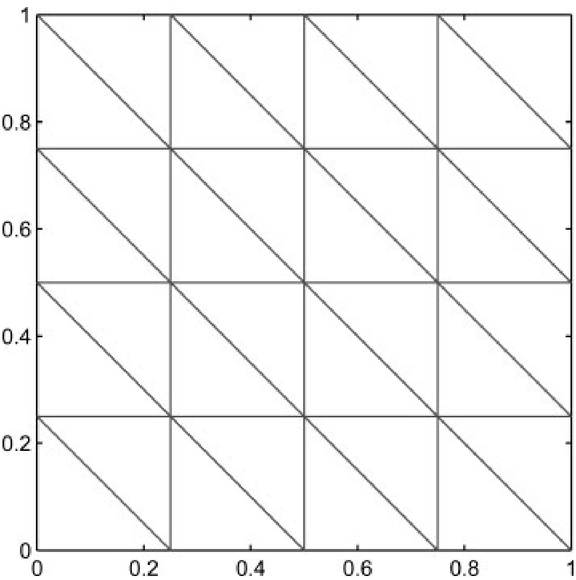

FIG. 4. The meshes with triangular elements (Example 4.2).

where $\tau_1 = 0.4$, $\tau_2 = 0.6$, and the matching points are changed accordingly. It is obvious that the method on non-Gaussian collocation points and non-Gaussian matching points is less accurate.

Example 4.2. We consider the same equation as in Example 4.1. Here we use a regular triangular partition, which is shown in Fig. 4. The nonconforming spline collocation method with $(n_c, n_m) = (1, 3)$ and $(n_c, n_m) = (6, 9)$ are applied for solving the equation (3.1). Numerical errors are presented in Tables IV and V, respectively. In the case of $(n_c, n_m) = (1, 3)$, there are only three unknowns at each element. Numerical results in Tables IV show that the accuracy of the method is in the order of $O(h^2)$ for the norms of $\|\cdot\|_{L^2}$, $\|\cdot\|_{L^\infty}$ and $\|\cdot\|_\infty$ and in the order of $O(h)$ for H^1 norm, which have the same accuracy as classical linear FEM method. Numerical results for

TABLE IV. The accuracy of the method with triangular elements $(n_c, n_m) = (1, 3)$ (Example 4.2).

error	$h = 1/8$	$h = 1/16$	$h = 1/32$	order (h^α)
$u = \sin(\pi x)\sin(\pi y)$				
$\|e\|_{L^2}$	9.248D-3	2.318D-3	5.797D-4	2.00
$\|e\|_{H^1}$	2.517D-1	1.259D-1	6.295D-2	1.00
$\|e\|_{L^\infty}$	3.794D-2	9.604D-3	2.408D-3	2.00
$\|e\|_\infty$	4.231D-3	1.068D-3	2.676D-4	2.00
$u = sin(x + 2y)$				
$\|e\|_{L^2}$	5.894D-3	1.488D-3	3.730D-4	2.00
$\|e\|_{H^1}$	1.227D-1	6.164D-2	3.086D-2	1.00
$\|e\|_{L^\infty}$	2.194D-2	5.590D-3	1.403D-3	1.99
$\|e\|_\infty$	5.055D-3	1.364D-3	3.730D-4	1.87
$u = 10e^x cos(y)$				
$\|e\|_{L^2}$	5.786D-2	1.471D-2	3.693D-3	1.99
$\|e\|_{H^1}$	1.178D0	5.961D-1	2.992D-1	0.99
$\|e\|_{L^\infty}$	2.655D-1	6.917D-2	1.759D-2	1.98
$\|e\|_\infty$	8.313D-2	2.316D-2	6.130D-3	1.92

TABLE V. The accuracy of the method with triangular elements $(n_c, n_m) = (6, 9)$ (Example 4.2).

error	$h = 1/4$	$h = 1/8$	$h = 1/16$	order (h^α)
$u = \sin(\pi x)\sin(\pi y)$				
$\|e\|_{L^2}$	7.605D-5	2.296D-6	7.081D-8	5.02
$\|e\|_{H^1}$	2.043D-3	1.254D-4	7.784D-6	4.01
$\|e\|_{L^\infty}$	2.727D-4	8.347D-6	2.591D-7	5.01
$\|e\|_\infty$	4.218D-5	1.271D-6	3.926D-8	5.02
$u = sin(x + 2y)$				
$\|e\|_{L^2}$	3.486D-7	1.124D-8	3.562D-10	4.98
$\|e\|_{H^1}$	1.586D-5	1.017D-6	6.424D-8	3.98
$\|e\|_{L^\infty}$	2.670D-6	8.538D-8	2.750D-9	4.96
$\|e\|_\infty$	3.999D-7	1.464D-8	4.749D-10	4.95
$u = 10e^x cos(y)$				
$\|e\|_{L^2}$	2.434D-6	7.465D-8	2.310D-9	5.01
$\|e\|_{H^1}$	1.220D-4	7.546D-6	4.689D-7	4.01
$\|e\|_{L^\infty}$	2.418D-5	8.262D-7	2.709D-8	4.93
$\|e\|_\infty$	2.631D-6	9.088D-8	3.050D-9	4.90

$(n_c, n_m) = (6, 9)$ exhibit that the method produces the accuracy $O(h^5)$ in the norms of $\|\cdot\|_{L^2}$, $\|\cdot\|_{L^\infty}$ and $\|\cdot\|_\infty$ and $O(h^4)$ in H^1 norm, which have the same order as classical fourth-order FEM method. The freedom of the nonconforming spline collocation method at each element is 9.

Example 4.3 (compressible flow). Next we consider the symmetric flow of an inviscid irrotational compressible fluid over a 6% thick circular airfoil. In the dimensionless form, this flow is described by

$$\left(1 - M_\infty^2 - M_\infty^2(1+\gamma)\frac{\partial\phi}{\partial x}\right)\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0, \tag{4.1}$$

where ϕ is the perturbed velocity potential, M_∞ the free stream Mach number, and γ the ratio of specific heats. The boundary conditions are given by

$$\frac{\partial\phi}{\partial n} = g'(x) \qquad \text{on the airfoil,}$$
$$\frac{\partial\phi}{\partial n} = 0 \qquad \text{on the line of symmetry,}$$
$$\phi = 0 \qquad \text{at infinity ,}$$

where the function $g(x)$ describes the geometry of the circular airfoil.

Numerical simulation of the compressible flow was made in [20] by using Galerkin method and least squares method in [4] by an orthogonal spline collocation method with rectangular meshes, where the computational domain is a rectangle under certain assumptions. The experimental measurement was also made by Kenchtel [21] at NASA. We apply the nonconforming spline collocation method for solving this flow problem in a nonrectangular domain.

Due to the nonlinearity of equation, an iterative scheme is defined by

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = M_\infty^2\left(1 + (1+\gamma)\frac{\partial\phi^0}{\partial x}\right)\frac{\partial^2\phi^0}{\partial x^2},$$

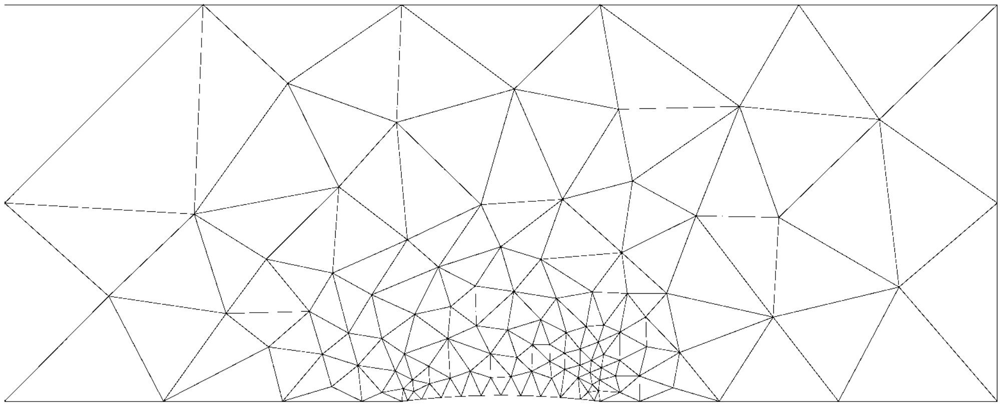

FIG. 5. The mesh for the compressible flow.

where ϕ^0 is the initial guess. The nonconforming spline collocation method with $(n_c, n_m) = (1, 3)$ are used to solve the flow problem. The computational domain and a triangular mesh are given in Fig. 5, in which more elements are placed around the airfoil. The convergence criterion for the iterative process is that the relative change in the local Mach number between two consecutive iterations should be less than a prescribed value ε, *i.e.*,

$$\max_{\text{all points}} \left| \frac{M^{(k+1)} - M^{(k)}}{M^{(k)}} \right| \leq \varepsilon, \tag{4.2}$$

where $M = V/a$ is the local Mach number, $V = ((1+\phi_x)^2 + \phi_y^2)^{1/2}$ is the local speed, and a the local speed of sound calculated from

$$a^2 = (\gamma - 1)\left(-\frac{1}{2}V^2 + \frac{1}{2} + \frac{1}{(\gamma - 1)M_\infty^2}\right). \tag{4.3}$$

Mach numbers of $M_\infty = 0.806$ and 0.840, all within the shock-free regime, are considered. The guessed value of $\phi^0 = 0$ is used throughout the domain. The iteration is terminated when the criterion is met with $\varepsilon = 10^{-3}$. The pressure coefficient plot of $C_p = -2u$ for these two different Mach numbers is given in Fig. 6. In the plot, the experimental measurements done in [21] are compared with the collocation results. The convergence is achieved in 17 iterations for both cases.

Example 4.4 (Elastic plate). Finally, we examine a two-dimensional benchmark problem in linear elasticity, a thin square elastic plate of the dimension 6×6 with a hole of the radius $r = 1.41$. The plate is stretched at the top $(y = 3)$ and at the bottom $(y = -3)$ by a constant tension and traction free is in the rest of the boundary. Due to the symmetry, only a quarter of the plate is described in Fig. 7. Here we assume that the plate is a homogeneous isotropic elastic body with $E = 2900$, $\nu = 0.4$ and plane strain conditions [22].

The nonconforming spline collocation method with $(n_c, n_m) = (6, 9)$ is applied for solving elasticity equations. The computational domain and two triangular meshes are given in Fig. 8, in which more elements are placed around the circular boundary.

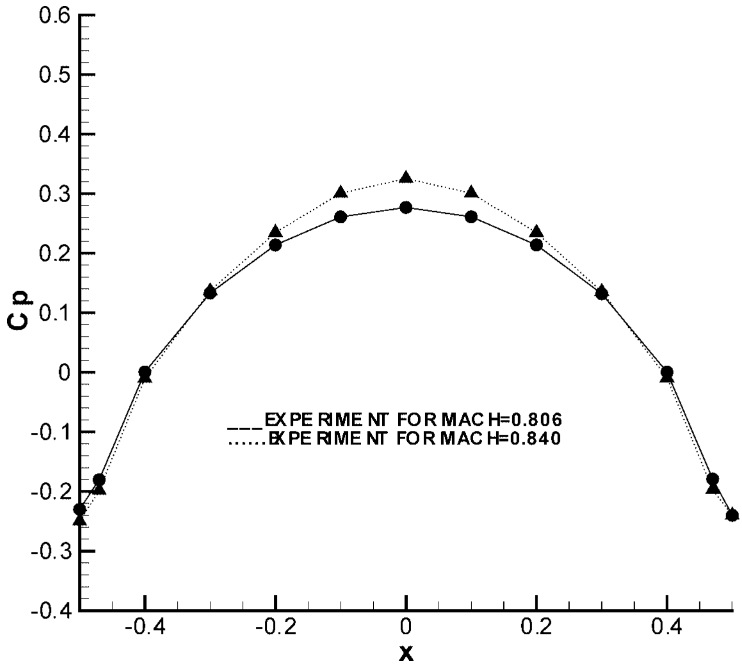

FIG. 6. Pressure coefficient along the airfoil.

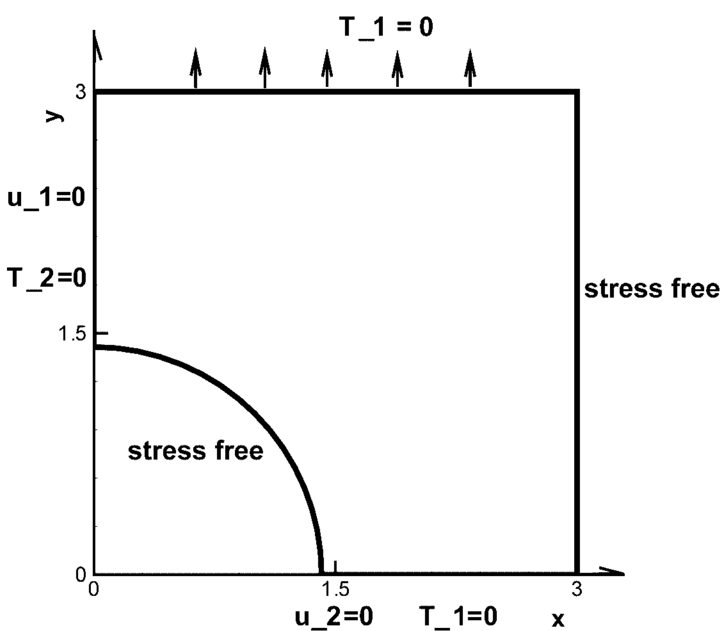

FIG. 7. An elastic plate with a circular hole.

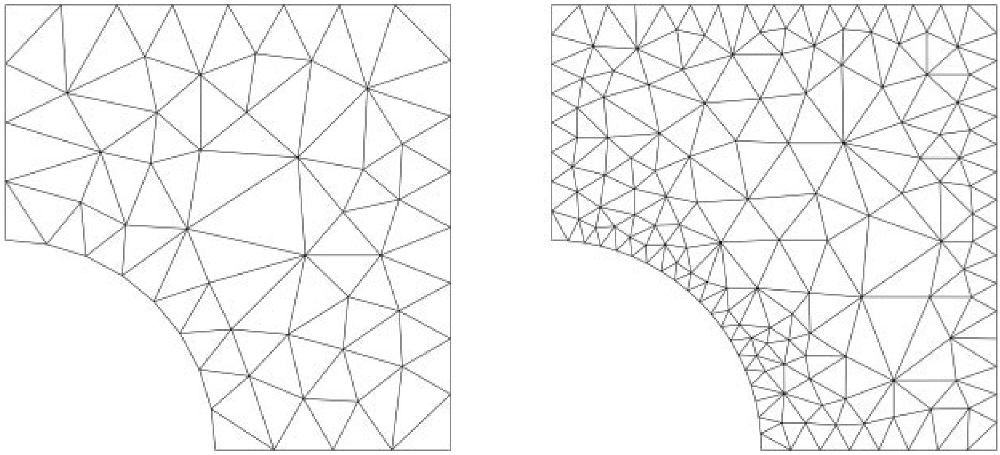

FIG. 8. The mesh for elastic plate.

Numerical results of displacements at some special points are presented in Table VI, compared with results obtained by a standard FEM code [22] with a combination of triangular and quadrilateral elements and 1122 degrees of freedom. Our method is of higher order accuracy.

V. CONCLUDING REMARKS

We have reformulated the spline collocation method proposed in [12] and show theoretically the existence and the optimal error estimate of numerical solution on a rectangular mesh (element). Our numerical simulations illustrate that the method is of high-order accuracy and ease of implementation. Based on our formulation, analysis and numerical results, we have observed many interesting features.

In the nonconforming spline collocation method, the numerical solution is re-expressed in (2.3), by using the local elimination (condensation) technique. The differential operator L has been implicitly embedded in the basis functions $\bar{\phi}_i$. The discrete system (2.4) is obtained by enforcing the solution being continuous on matching points. The relationship between spline collocation methods, finite difference methods and finite element methods have been studied by many authors. This nonconforming spline collocation method could be viewed as a numerical method which has a good connection with both finite difference method and finite element method. In our analysis, the nonconforming spline collocation method on a rectangular mesh is equivalent to a special high-order finite difference method. The truncation error and the error in a energy norm are obtained in Lemma 3.2, in terms of a typical finite difference approach. However a standard

TABLE VI. The comparison of numerical results (Example 4.4).

Methods	FEM [22]	collocation (mesh I)	collocation (mesh II)
$u_1\,(1.41, 0)$	−1.618e-3	−1.614e-3	−1.626e-3
$u_2\,(0, 1.41)$	2.648e-3	2.651e-3	2.665e-3
$u_1\,(3, 1.5)$	−1.210e-3	−1.217e-3	−1.220e-3
$u_2\,(1.5, 3)$	1.858e-3	1.858e-3	1.872e-3

finite element analysis, Green's formula, is applied to obtain the optimal error estimate in (3.17) and Theorem 3.5. On the other hand, classical finite element method is based on the Green's formula. Due to the nature of the nonconforming spline spaces, we have

$$\sum_{e_k \in \Pi_h} \int_{e_k} v \Delta w \, dxdy = \sum_{e_k \in \Pi_h} \int_{\partial e_k} \frac{\partial w}{\partial n} v \, ds - \sum_{e_k \in \Pi_h} \int_{e_k} \nabla w \cdot \nabla v \, dxdy. \tag{5.1}$$

When $w = u_I - u_h$,

$$\Delta w(z_{c,i}) = 0,$$

and both w and $\frac{\partial w}{\partial n}$ are continuous on matching points. This implies that the left side of (5.1) depends upon the collocation points and the first term in the right side of (5.1) is related to the matching points. Moreover, in some cases, such as the triangular elements with $(n_c, n_m) = (1, 3)$ and $(n_c, n_m) = (6, 9)$ and the rectangular element studied in section 3, $\Delta w = 0$. In addition, the collocation system defined in (2.4) seems much different from those finite difference systems and finite element systems. Our analysis in section 3 shows the equivalence between a spline collocation system and a special finite difference system. Also one can see from (5.1) some relationship between the collocation system and FEM system. The error analysis of nonconforming finite element methods with either triangular and quadrilateral meshes has been well done [23–25], where the error function on the interface of elements can be estimated. However, the spline collocation is based on a discrete form of (5.1). The error estimate on the interface of elements is difficult in the general case.

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