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The superconvergence of the composite midpoint rule for the finite-part integral

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ABSTRACT

The composite midpoint rule is probably the simplest one among the Newton–Cotes rules for Riemann integral. However, this rule is divergent in general for Hadamard finite-part integral. In this paper, we turn this rule to a useful one and, apply it to evaluate Hadamard finite-part integral as well as to solve the relevant integral equation. The key point is based on the investigation of its *pointwise superconvergence phenomenon*, i.e., when the singular point coincides with some *a priori* known point, the convergence rate of the midpoint rule is higher than what is globally possible. We show that the superconvergence rate of the composite midpoint rule occurs at the midpoint of each subinterval and obtain the corresponding superconvergence error estimate. By applying the midpoint rule to approximate the finite-part integral and by choosing the superconvergence points as the collocation points, we obtain a collocation scheme for solving the finite-part integral equation. More interesting is that the inverse of the coefficient matrix of the resulting linear system has an explicit expression, by which an optimal error estimate is established. Some numerical examples are provided to validate the theoretical analysis.

1. Introduction

We consider the finite-part integral equation of the form

$$\int\!\!\!\!\!\!= _a^b \frac{u(x)}{(x-s)^2}\mathrm{d}x = f(s), \tag{1.1}$$

where $f(s)$ is a given function. The integral in the left hand side must be understood in the Hadamard finite-part sense. Integrals of this kind have many different but equivalent definitions, among which we prefer the definition (see, e.g., [1–3])

$$\int\!\!\!\!\!\!= _a^b \frac{u(x)}{(x-s)^2}\mathrm{d}x := \lim_{\varepsilon\to 0}\left\{\int_a^{s-\varepsilon}\frac{u(x)}{(x-s)^2}\mathrm{d}x + \int_{s+\varepsilon}^{b}\frac{u(x)}{(x-s)^2}\mathrm{d}x - \frac{2u(s)}{\varepsilon}\right\}, \tag{1.2}$$

where $s \in (a, b)$. $u(x)$ is said to be finite-part integrable with respect to the weight $(x-s)^{-2}$ if the limit on the right hand side of the above equation exists. A sufficient condition for $u(x)$ to be finite-part integrable is that its first derivative $u'(x)$

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is Hölder continuous. This regularity condition can be weaken to some extent [4]. An extension of (1.2) can be found in a recent paper [5]. Recall that the Cauchy principal value integral or the finite Hilbert transform is defined by

$$\fint_a^b \frac{u(x)}{x-s}\mathrm{d}x := \lim_{\varepsilon\to 0}\left\{\int_a^{s-\varepsilon}\frac{u(x)}{x-s}\mathrm{d}x + \int_{s+\varepsilon}^b \frac{u(x)}{x-s}\mathrm{d}x\right\}, \quad s\in(a,b). \tag{1.3}$$

Obvious, definite (1.2) is a natural extension of (1.3). The Hadamard finite-part integral can be related to the Cauchy principal value integral in two ways. First, the former is viewed as the derivative, with respective to the singular point s, of the later, i.e.,

$$\oint_a^b \frac{u(x)}{(x-s)^2}\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}s}\left(\fint_a^b \frac{u(x)}{x-s}\mathrm{d}x\right). \tag{1.4}$$

In many occasions, this identity has been used as an alternative definition of the Hadamard finite-part integral (cf. [6–9]). Secondly, both kinds of singular integrals are related via the formula,

$$\oint_a^b \frac{u(x)}{(x-s)^2}\mathrm{d}x = \frac{u(a)}{a-s} - \frac{u(b)}{b-s} + \fint_a^b \frac{u'(x)}{x-s}\mathrm{d}x, \tag{1.5}$$

which is obtained directly from (1.2) and (1.3). (1.5) can be viewed as the formula of integration by parts for the finite-part integral. Throughout this paper, $\oint$ denotes a finite-part integral and by contrast, $\fint$ a Cauchy principal value integral.

Finite-part integrals arise frequently in boundary element methods (BEMs) and other numerical computations. This kind of integrals can be evaluated numerically by a number of methods, such as the Gaussian method [10,9,11–14], the (composite) Newton–Cotes method [15,1,16,3], the transformation method [6,7] and some others [8,17]. Generally speaking, Gaussian method and the transformation method are very efficient if the integrand function is smooth enough. Otherwise, if the integrand has a lower regularity or the values of the integrand are only tabulated at a number of specified nodes, the Newton–Cotes method becomes competitive.

The Newton–Cotes method for Hadamard finite-part integral is a natural extension of its counterpart for Riemann integral. The most characteristic of this method is that it has a flexibility in selecting the quadrature nodes and the piecewise interpolant. Usually, the quadrature nodes in Newton–Cotes method are equally spaced over the integral interval. If the integrand function is approximated by a piecewise polynomial interpolant of degree k ($k \geq 1$), the accuracy of the Newton–Cotes method associated with the usual Riemann integrals is $O(h^{k+1})$ for odd k and $O(h^{k+2})$ for even k. However, due to the hypersingular singularity of the integrand at the singular point s, the correspondent result for Hadamard finite-part integral (1.2) is generally only $O(h^k)$ [3]. When the singular points s coincides with some *a priori* known point, the accuracy can reach $O(h^{k+1})$. We refer to this as *the pointwise superconvergence phenomenon* of the Newton–Cotes method for the finite-part integrals.

The superconvergence of Newton–Cotes rules for Hadamard finite-part integrals was first studied in [18,19], where the superconvergence rate of the composite trapezoidal rule ($k = 1$) and the composite Simpson's rule ($k = 2$) was presented, respectively. Recently, a general result was given in [3] for $k \geq 1$. The composite midpoint rule ($k = 0$), one of the lowest order Newton–Cotes rules, is widely used in the evaluation of integrals with smooth, weakly or Cauchy singular integrands due to its simplicity and ease of implementation. However, this rule is divergent in general for finite-part integrals, which is probably the reason that it has drawn so little attention. In this paper, we show that, in despite of its global divergence, the composite midpoint rule can still be used in the evaluation of finite-part integrals. The key point is based on the investigation of its superconvergence.

Up to now, the application of the superconvergence results has been mainly confined to the numerical evaluation of finite-part integrals. However, from the applications point of view, it is more interesting to apply the superconvergence results to the numerical solution of finite-part integral equations arising in various applied sciences and engineering problems. In some recent papers, the problem of applying the superconvergence results to the solution of finite-part equations was discussed, however, no error estimate was presented [5,20,21]. In this paper, we use the superconvergence result of the midpoint rule to construct a collocation scheme for solving the finite-part integral equation and obtain, for the first time, an optimal error estimate.

The rest of this paper is organized as follows. In Section 2, we construct the composite midpoint rule for the finite-part integral by employing the piecewise constant interpolation. Then we demonstrate by an example that this rule is divergent in general. In Section 3, we suggest a modified midpoint rule and obtain its error estimate, by which the superconvergence result of the original midpoint rule is established. In Section 4, we present a collocation scheme for solving a certain finite-part integral equation. Based on the superconvergence result, we obtain an optimal error estimate for the collocation scheme. Finally, we present in the last section several numerical examples to validate the analysis as well as to show the efficiency of our method.

2. The composite midpoint rule

For simplicity of exposition, we confine ourselves to the case where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ is a uniform mesh of $[a, b]$ with mesh size $h = (b - a)/n$. It is not very difficult to extend our results to certain quasi-uniform mesh. Let $u_0^I(x)$

be the piecewise constant interpolation of $u(x)$, defined by

$$u_0^I(x) = \sum_{i=1}^{n} u(\hat{x}_i)\varphi_i(x), \tag{2.1}$$

where $\hat{x}_i = (x_{i-1} + x_i)/2$ and

$$\varphi_i(x) = \begin{cases} 1, & \text{on } [x_{i-1}, x_i], \\ 0, & \text{otherwise.} \end{cases}$$

Replacing $u(x)$ in the finite-part integral of (1.1) with $u_0^I(x)$ gives the following composite midpoint rule

$$\mathcal{Q}_{n,0}(u) := \mathop{=\!\!\!\!\!\!\int}_a^b \frac{u_0^I(x)}{(x-s)^2}\mathrm{d}x = \sum_{i=1}^{n} \omega_i^{(0)}(s)u(\hat{x}_i) = \mathop{=\!\!\!\!\!\!\int}_a^b \frac{u(x)}{(x-s)^2}\mathrm{d}x - \mathcal{E}_{n,0}(u;s), \tag{2.2}$$

where $\mathcal{E}_{n,0}(u;s)$ denotes the error functional and

$$\omega_i^{(0)}(s) = \mathop{=\!\!\!\!\!\!\int}_a^b \frac{\varphi_i(x)}{(x-s)^2}\mathrm{d}x = \frac{1}{x_{i-1}-s} - \frac{1}{x_i - s}.$$

Clearly, the above composite midpoint rule is exact when the integrand function $u(x)$ is a constant function. However, this quadrature rule is almost useless since it is divergent in general. In order to drive this point clear, let us first introduce some notations and identities. Let $Q_k(x)$ be the second function associated with the Legendre polynomial $P_k(x)$, defined by

$$Q_k(x) = \begin{cases} \dfrac{1}{2} \displaystyle{-\!\!\!\!\!\!\int}_{-1}^{1} \dfrac{P_k(t)}{x-t}\mathrm{d}t, & |x| < 1, \\ \dfrac{1}{2} \displaystyle\int_{-1}^{1} \dfrac{P_k(t)}{x-t}\mathrm{d}t, & |x| > 1. \end{cases} \tag{2.3}$$

For the above defined functions, we have (see, e.g, [22])

$$\begin{aligned} &Q_0(x) = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|, \quad Q_1(x) = xQ_0(x) - 1, \\ &Q_{k+1}(x) = \frac{2k+1}{k+1}xQ_k(x) - \frac{k}{k+1}Q_{k-1}(x), \quad k = 1, 2, \ldots. \end{aligned} \tag{2.4}$$

By the classical identity [22]

$$Q_k(x) = \frac{1}{2^{k+1}}\int_{-1}^{1} \frac{(1-t^2)^k}{(x-t)^{k+1}}\mathrm{d}t, \quad |x| > 1,\ k = 0, 1, 2, \ldots, \tag{2.5}$$

we get

$$\left|Q_k'(x)\right| = \frac{k+1}{2^{k+1}}\left|\int_{-1}^{1} \frac{(1-t^2)^k}{(x-t)^{k+2}}\mathrm{d}t\right| \le \frac{C}{(|x|-1)^{k+2}}, \quad |x| > 1. \tag{2.6}$$

In the following, C will denote a generic positive constant which is independent of the mesh parameter h and the singular point s. Finally, define the operator $\mathcal{W}$ by

$$\mathcal{W}(f;\tau) := f(\tau) + \sum_{i=1}^{\infty}[f(2i+\tau) + f(-2i+\tau)], \quad \tau \in (-1, 1). \tag{2.7}$$

Obvious, $\mathcal{W}$ is a linear operator on $f(x)$. By direct computation,

$$\begin{aligned} \mathcal{W}(Q_0;\tau) &= \frac{1}{2}\ln\frac{1+\tau}{1-\tau} + \frac{1}{2}\sum_{i=1}^{\infty}\left(\ln\frac{2i+1+\tau}{2i-1+\tau} + \ln\frac{2i-1-\tau}{2i+1-\tau}\right) \\ &= \lim_{i\to\infty}\frac{1}{2}\ln\frac{2i+1+\tau}{2i+1-\tau} = 0, \\ \mathcal{W}(xQ_0';\tau) &= \frac{\tau}{1-\tau^2} - \sum_{i=1}^{\infty}\left[\frac{2i+\tau}{(2i+\tau)^2-1} + \frac{-2i+\tau}{(-2i+\tau)^2-1}\right] \\ &= \sum_{i=1}^{\infty}\left(\frac{1}{2i-1-\tau} + \frac{1}{-2i+1-\tau}\right) \\ &= \frac{1}{2}\lim_{n\to\infty}\sum_{k=-n}^{n}\frac{1}{k+\frac{1}{2}-\frac{\tau}{2}} \\ &= \frac{\pi}{2}\tan\frac{\tau\pi}{2}, \end{aligned}$$

where we have used the well known identity (cf. [23], (1.2.7))

$$\lim_{n\to\infty}\sum_{k=-n}^{n}\frac{1}{k+\frac{1}{2}-x}=\pi\tan(\pi x). \tag{2.8}$$

Then, it follows that

$$\mathcal{W}(Q_1';\tau)=\mathcal{W}(Q_0+xQ_0';\tau)=\frac{\pi}{2}\tan\frac{\tau\pi}{2}. \tag{2.9}$$

Now, we are ready to explain why the composite midpoint rule is divergent in general. Consider the special case $u(x)=x$ and assume $s=x_{m-1}+(\tau+1)h/2$, where $\tau\in(-1,1)$ is the local coordinate of s. By (2.9) and the definition of $\mathcal{W}$, we find

$$\begin{aligned}\mathcal{E}_{n,0}(x;s)&=\left(\fint_a^b+\sum_{i=1,i\neq m}^{n}\int_{x_{i-1}}^{x_i}\right)\frac{x-\hat{x}_i}{(x-s)^2}\mathrm{d}x\\&=\sum_{i=1}^{n}\left[\ln\left|\frac{x_i-s}{x_{i-1}-s}\right|+\frac{(s-\hat{x}_i)h}{(x_i-s)(x_{i-1}-s)}\right]\\&=-2\sum_{i=1}^{n}Q_1'(2(m-i)+\tau)\\&=-\pi\tan\frac{\tau\pi}{2}+2\sum_{i=m}^{\infty}Q_1'(2i+\tau)+2\sum_{i=n-m+1}^{\infty}Q_1'(-2i+\tau).\end{aligned} \tag{2.10}$$

If $s\in[a+\delta,b-\delta]$ where δ is a positive constant, independent of n, then m and $n-m+1$ will tend to infinity as $n\to\infty$. Thus, by (2.6),

$$\lim_{n\to\infty}\left[\sum_{i=m}^{\infty}Q_1'(2i+\tau)+\sum_{i=n-m+1}^{\infty}Q_1'(-2i+\tau)\right]=0$$

and as a consequent,

$$\lim_{n\to\infty}\mathcal{E}_{n,0}(s;x)=-\pi\tan\frac{\tau\pi}{2}. \tag{2.11}$$

It is now clear that $\mathcal{E}_{n,0}(u;s)$ does not vanish in general when n approaches infinity. Hence the composite midpoint rule cannot be used directly in practical computation before some special treatments have been made.

3. The superconvergence result

In the above section, we have seen that the composite midpoint rule is generally divergent. In order to build this rule into a useful one, we suggest two approaches. In the first approach we modify the composite midpoint rule slightly to get a new rule and in the second approach, we utilize the superconvergence property of the composite midpoint rule.

First, we suggest a modified composite midpoint rule $\widetilde{\mathcal{Q}}_{n,0}(u)$, defined by,

$$\widetilde{\mathcal{Q}}_{n,0}(u)=\mathcal{Q}_{n,0}(u)-\pi u'(s)\tan\frac{\tau\pi}{2}, \tag{3.1}$$

where τ is the local coordinate of the singular point s. For this modified quadrature rule, we have the following error estimate.

Theorem 3.1. *Assume that $u(x)\in C^{2+\alpha}[a,b](0<\alpha\le 1)$ and $s\neq x_i$ for any $i=1,2,\dots,n-1$. Then, for the modified composite midpoint rule defined by (3.1) and (2.2), there holds the error estimate*

$$\left|\fint_a^b\frac{u(x)}{(x-s)^2}\mathrm{d}x-\widetilde{\mathcal{Q}}_{n,0}(u)\right|\le\begin{cases}C[\gamma^{-1}(h,s)+\eta^2(s)h^{1-\alpha}]h^{1+\alpha}, & 0<\alpha<1,\\ C[\gamma^{-1}(h,s)+|\ln h|+\eta^2(s)]h^2, & \alpha=1,\end{cases} \tag{3.2}$$

where

$$\begin{aligned}\gamma(h,s)&=\min_{0\le i\le n}\frac{|s-x_i|}{h},\\ \eta(s)&=\max\left\{\frac{1}{b-s},\frac{1}{s-a}\right\}.\end{aligned} \tag{3.3}$$

Proof. Assume that $s = x_{m-1} + (\tau + 1)h/2$ with $1 \le m \le n$ and $\tau \in (-1, 1)$. Let $u_q^I(x)$ be the piecewise quadratic interpolation function of $u(x)$, defined by

$$u_q^I(x) = u(x_{i-1})\frac{2(x-x_i)(x-\hat{x}_i)}{h^2} - u(\hat{x}_i)\frac{4(x-x_i)(x-x_{i-1})}{h^2} + u(x_i)\frac{2(x-x_{i-1})(x-\hat{x}_i)}{h^2}, \quad x \in [x_{i-1}, x_i].$$

Then, the error can be split into two parts,

$$\mathop{\ne\!\!\!\!\!\!\int}_a^b \frac{u(x)}{(x-s)^2}\mathrm{d}x - \widetilde{\mathcal{Q}}_{n,0}(u) = \mathop{\ne\!\!\!\!\!\!\int}_a^b \frac{u(x)-u_q^I(x)}{(x-s)^2}\mathrm{d}x + \left[\left(\mathop{\ne\!\!\!\!\!\!\int}_{x_{m-1}}^{x_m} + \sum_{i=1,i\ne m}^{n}\int_{x_{i-1}}^{x_i}\right)\frac{u_q^I(x)-u(\hat{x}_i)}{(x-s)^2}\mathrm{d}x + \pi u'(s)\tan\frac{\tau\pi}{2}\right]. \tag{3.4}$$

For the first part, we have the following estimate (cf. [19], Lemma 3.3)

$$\left|\mathop{\ne\!\!\!\!\!\!\int}_a^b \frac{u(x)-u_q^I(x)}{(x-s)^2}\mathrm{d}x\right| \le C\gamma^{-1}(h,s)h^{1+\alpha}, \quad u(x) \in C^{2+\alpha}[-1,1], \tag{3.5}$$

where $\gamma(h, s)$ is defined in (3.3). Thus, the nontrivial part is to estimate the second term in the right hand side of (3.4). By definition,

$$u_q^I(x) - u(\hat{x}_i) = \alpha_i(x-\hat{x}_i)^2 + \beta_i(x-\hat{x}_i),$$

where

$$\begin{aligned}
\alpha_i &= \frac{2u(x_{i-1}) + 2u(x_i) - 4u(\hat{x}_i)}{h^2} \\
&= \frac{u''(\eta_i) + u''(\zeta_i) - 2u''(\hat{x}_i)}{4} + \frac{u''(\hat{x}_i) - u''(s)}{2} + \frac{u''(s)}{2}, \\
\beta_i &= \frac{u(x_i) - u(x_{i-1})}{h} \\
&= \frac{u''(\eta_i) - u''(\zeta_i)}{8}h + [u'(\hat{x}_i) - u'(s)] + u'(s)
\end{aligned}$$

and $\eta_i, \zeta_i \in (x_{i-1}, x_i)$. By straightforward calculation, we have

$$\begin{aligned}
&\left(\mathop{\ne\!\!\!\!\!\!\int}_{x_{m-1}}^{x_m} + \sum_{i=1,i\ne m}^{n}\int_{x_{i-1}}^{x_i}\right)\frac{u_q^I(x)-u(\hat{x}_i)}{(x-s)^2}\mathrm{d}x + \pi u'(s)\tan\frac{\tau\pi}{2} \\
&= \left(\mathop{\ne\!\!\!\!\!\!\int}_{x_{m-1}}^{x_m} + \sum_{i=1,i\ne m}^{n}\int_{x_{i-1}}^{x_i}\right)\frac{\alpha_i(x-\hat{x}_i)^2 + \beta_i(x-\hat{x}_i)}{(x-s)^2}\mathrm{d}x + \pi u'(s)\tan\frac{\tau\pi}{2} \\
&= \sum_{i=1}^{n}\alpha_i\left[h + 2(s-\hat{x}_i)\ln\left|\frac{x_i-s}{x_{i-1}-s}\right| + \frac{h(s-\hat{x}_i)^2}{(x_i-s)(x_{i-1}-s)}\right] \\
&\quad + \sum_{i=1}^{n}\beta_i\left[\ln\left|\frac{x_i-s}{x_{i-1}-s}\right| + \frac{h(s-\hat{x}_i)}{(x_i-s)(x_{i-1}-s)}\right] + \pi u'(s)\tan\frac{\tau\pi}{2} \\
&= -\frac{h}{3}\sum_{i=1}^{n}\alpha_i[2Q_2'(2(m-i)+\tau) + Q_0'(2(m-i)+\tau)] - 2\sum_{i=1}^{n}\beta_i Q_1'(2(m-i)+\tau) + \pi u'(s)\tan\frac{\tau\pi}{2} \\
&:= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4 + \mathcal{I}_5 + \mathcal{I}_6,
\end{aligned}$$

where

$$\mathcal{I}_1 = -\frac{h}{3}\sum_{i=1}^{n}\frac{u''(\eta_i) + u''(\zeta_i) - 2u''(\hat{x}_i)}{4}[2Q_2'(2(m-i)+\tau) + Q_0'(2(m-i)+\tau)],$$

$$\mathcal{I}_2 = -\frac{h}{3}\sum_{i=1}^{n}\frac{u''(\hat{x}_i) - u''(s)}{2}[2Q_2'(2(m-i)+\tau) + Q_0'(2(m-i)+\tau)],$$

$$\mathcal{I}_3 = -\frac{u''(s)h}{6}\sum_{i=1}^{n}[2Q_2'(2(m-i)+\tau) + Q_0'(2(m-i)+\tau)],$$

$$\mathcal{I}_4 = -2h\sum_{i=1}^{n}\frac{u''(\eta_i) - u''(\zeta_i)}{8}Q_1'(2(m-i)+\tau),$$

$$\mathcal{I}_5 = -2\sum_{i=1}^{n}[u'(\hat{x}_i) - u'(s)]Q_1'(2(m-i)+\tau),$$

$$\mathcal{I}_6 = -2u'(s)\left[\sum_{i=1}^{n} Q_1'(2(m-i)+\tau) - \frac{\pi}{2}\tan\frac{\tau\pi}{2}\right].$$

We now estimate $\mathcal{I}_i (1 \le i \le 6)$ one by one. From (2.6), we have

$$\begin{aligned}\sum_{i=1}^{n}|2(m-i)+\tau|^\alpha \left|Q_k'(2(m-i)+\tau)\right| &\le |\tau|^\alpha |Q_k'(\tau)| + C\sum_{i=1, i\neq m}^{n}\frac{|2(m-i)+\tau|^\alpha}{(|2(m-i)+\tau|-1)^{k+2}} \\ &\le \begin{cases} C, & 0<\alpha<1, \\ C, & \alpha = 1, k>0, \\ C|\ln h|, & \alpha=1,\ k=0 \end{cases}\end{aligned}$$

and

$$\sum_{i=1}^{n}\left|Q_k'(2(m-i)+\tau)\right| \le |Q_k'(\tau)| + C\sum_{i=1, i\neq m}^{n}\frac{1}{(|2(m-i)+\tau|-1)^{k+2}} \le C.$$

By these inequalities, we obtain at once the estimates,

$$|\mathcal{I}_1| \le Ch^{1+\alpha}, \qquad |\mathcal{I}_4| \le Ch^{1+\alpha} \tag{3.6}$$

and

$$|\mathcal{I}_2| \le \begin{cases} Ch^{1+\alpha}, & 0<\alpha<1, \\ C|\ln h|h^2, & \alpha=1. \end{cases} \tag{3.7}$$

Using the identity,

$$2Q_2'(x) + Q_0'(x) - 6xQ_1'(x) = \frac{3}{x^2-1},$$

we get

$$\begin{aligned}\mathcal{I}_3 - 2u''(s)\sum_{i=1}^{n}(\hat{x}_i - s)Q_1'(2(m-i)+\tau) &= -\frac{u''(s)h}{6}\sum_{i=1}^{n}\{2Q_2'(2(m-i)+\tau) + Q_0'(2(m-i)+\tau) \\ &\quad - 6[2(m-i)+\tau]Q_1'(2(m-i)+\tau)\} \\ &= -\frac{u''(s)h}{2}\sum_{i=1}^{n}\frac{1}{[2(m-i)+\tau]^2-1} \\ &= \frac{u''(s)h}{4}\left[\frac{1}{2m-1+\tau} + \frac{1}{2(n-m)+1-\tau}\right],\end{aligned}$$

which implies

$$\left|\mathcal{I}_3 - 2u''(s)\sum_{i=1}^{n}(\hat{x}_i - s)Q_1'(2(m-i)+\tau)\right| \le C\eta(s)h^2. \tag{3.8}$$

Moreover,

$$\begin{aligned}\left|\mathcal{I}_5 + 2u''(s)\sum_{i=1}^{n}(\hat{x}_i - s)Q_1'(2(m-i)+\tau)\right| &= \left|2\sum_{i=1}^{n}[u''(s) - u''(\bar{x}_i)](\hat{x}_i - s)Q_1'(2(m-i)+\tau)\right| \\ &\le C\sum_{i=1}^{n}|\hat{x}_i - s|^{1+\alpha}\left|Q_1'(2(m-i)+\tau)\right| \\ &\le Ch^{1+\alpha}\sum_{i=1}^{n}|2(m-i)+\tau|^{1+\alpha}\left|Q_1'(2(m-i)+\tau)\right| \\ &\le Ch^{1+\alpha}\left\{|\tau|^{1+\alpha}|Q_1'(\tau)| + \sum_{i=1, i\neq m}^{n}\frac{|2(m-i)+\tau|^{1+\alpha}}{[|2(m-i)+\tau|-1]^3}\right\} \\ &\le \begin{cases} Ch^{1+\alpha}, & 0<\alpha<1, \\ C|\ln h|h^2, & \alpha=1, \end{cases}\end{aligned} \tag{3.9}$$

where $\bar{x}_i \in (s, \hat{x}_i)$ or $(\hat{x}_i, s)$ and (2.6) has been used. Combining (3.9) with (3.8) and by the triangle inequality, we reach

$$|\mathcal{I}_3 + \mathcal{I}_5| \leq \begin{cases} C[1 + \eta(s)h^{1-\alpha}]h^{1+\alpha}, & 0 < \alpha < 1, \\ C[|\ln h| + \eta(s)]h^2, & \alpha = 1. \end{cases} \tag{3.10}$$

To estimate $\mathcal{I}_6$, we use (2.7) and (2.9) to get

$$\sum_{i=1}^{n} Q_1'(2(m-i)+\tau) - \frac{\pi}{2}\tan\frac{\tau\pi}{2} = -\sum_{i=m}^{\infty} Q_1'(2i+\tau) - \sum_{i=n-m+1}^{\infty} Q_1'(-2i+\tau)$$

and by (2.6)

$$\begin{aligned} |\mathcal{I}_6| &\leq C\left[\sum_{i=m}^{\infty}\frac{1}{(|2i+\tau|-1)^3} + \sum_{i=n-m+1}^{\infty}\frac{1}{(|2i-\tau|-1)^3}\right] \\ &\leq C\left[\frac{1}{m^2} + \frac{1}{(n-m+1)^2}\right] \leq C\eta^2(s)h^2. \end{aligned} \tag{3.11}$$

Putting together (3.6), (3.7), (3.10) and (3.11) and using the inequality

$$\eta(s) \leq (b-a)\eta^2(s),$$

we get

$$\left|\left(\int_{x_{m-1}}^{x_m} + \sum_{i=1, i\neq m}^{n}\int_{x_{i-1}}^{x_i}\right)\frac{u_q^I(x) - u(\hat{x}_i)}{(x-s)^2}\mathrm{d}x + \pi u'(s)\tan\frac{\tau\pi}{2}\right| \leq \begin{cases} C[1 + \eta^2(s)h^{1-\alpha}]h^{1+\alpha}, & 0 < \alpha < 1, \\ C[|\ln h| + \eta^2(s)]h^2, & \alpha = 1. \end{cases} \tag{3.12}$$

Combining the above estimate with (3.5) leads to (3.2), which completes the proof. □

Remark. The modified midpoint rule $\widetilde{\mathcal{Q}}_{n,0}(u)$ is still valid if the regularity assumption on $u(x)$ is a little weaker than that in Theorem 3.1. In fact, if $u(x) \in C^{1+\alpha}[a, b] (0 < \alpha \leq 1)$ and the other assumptions in Theorem 3.1 are kept unchanged, then we have the estimate

$$\left|\int_a^b \frac{u(x)}{(x-s)^2}\mathrm{d}x - \widetilde{\mathcal{Q}}_{n,0}(u)\right| \leq C[\gamma^{-1}(h, s) + \eta^2(s)h^{2-\alpha}]h^{\alpha}. \tag{3.13}$$

The proof is similar to that of Theorem 3.1, except that a piecewise linear interpolant, instead of $u_q^I(x)$, is employed.

When the singular point s coincides with a subinterval midpoint, its local coordinate τ vanishes and as a result, the modified composite midpoint rule degenerates to the original composite midpoint rule. Thus the superconvergence result follows immediately form Theorem 3.1.

Theorem 3.2. *Assume that $u(x) \in C^{2+\alpha}[a, b] (0 < \alpha \leq 1)$. Then, for the composite midpoint rule $\mathcal{Q}_{n,0}(u)$ defined by (2.2), there holds at $s = \hat{x}_m (1 \leq m \leq n)$ the superconvergence estimate*

$$\left|\int_a^b \frac{u(x)}{(x-s)^2}\mathrm{d}x - \mathcal{Q}_{n,0}(u)\right| \leq \begin{cases} C[1 + \eta^2(s)h^{1-\alpha}]h^{1+\alpha}, & 0 < \alpha < 1, \\ C[|\ln h| + \eta^2(s)]h^2, & \alpha = 1. \end{cases} \tag{3.14}$$

Based on Theorems 3.1 and 3.2, the composite midpoint rule is now useful in two respects. First, the modified composite midpoint rule $\widetilde{\mathcal{Q}}_{n,0}(u)$, instead of $\mathcal{Q}_{n,0}(u)$ itself, can be used to evaluate the finite-part integral. Second, by employing the superconvergence property, the rule $\mathcal{Q}_{n,0}(u)$ itself can also be used in practical computation, provided that the singular point s happens to be a superconvergence point. In the case where the singular point s is not a superconvergence point with respect to the uniform mesh $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, it is always possible for us to translate the interior mesh nodes $x_i (1 \leq i \leq n-1)$ so that the singular point is a superconvergence point of the new mesh. The subintervals in the new mesh are uniform except those two at the endpoints a and b. It is not difficult to extend the above superconvergence analysis to this quasi-uniform mesh and as a result, the composite midpoint rule is valid on this new mesh.

4. A collocation scheme for the finite-part integral equation

In this section, we use the superconvergence result to study a collocation scheme for the solution of the finite-part integral Eq. (1.1). By using the composite midpoint rule $\mathcal{Q}_{n,0}(u)$ to approximate the finite-part integral in the left hand side and by collocating the resulting equation at the superconvergence points $\hat{x}_i = x_{i-1} + h/2$ $(i = 1, 2, \ldots, n)$, we get the following linear system

$$\mathcal{A}_n\mathbf{U}_a = \mathbf{F}_e, \tag{4.1}$$

where

$$\mathcal{A}_n = (a_{ij})_{n\times n}, \quad a_{ij} = \frac{1}{x_{j-1} - \hat{x}_i} - \frac{1}{x_j - \hat{x}_i} = \frac{4}{[4(j-i)^2 - 1]h}, \qquad \mathbf{U}_a = (u_1, u_2, \ldots, u_n)^{\mathrm{T}}, \qquad \mathbf{F}_e = (f(\hat{x}_1), f(\hat{x}_2), \ldots, f(\hat{x}_n))^{\mathrm{T}} \tag{4.2}$$

and u_i $(i = 1, 2, \ldots, n)$ denotes the approximation of $u(x)$ at $\hat{x}_i$.

We mention that, the above collocation scheme has been successfully applied for solving a special finite-part integral equation of the form (1.1) with the right hand side function $f(s)$ containing a delta function and the unknown function $u(x)$ vanishing at the endpoints a and $b(a = -b = -1)$ (cf. [24]). The authors of [24] also obtained the convergence result of the scheme for that special case. However, the superconvergence result was neither mentioned nor studied by the authors, even in the recent papers [25,26]. The main point of this section is to obtain an error estimate for the collocation scheme (4.1) by using the superconvergence result of the composite midpoint rule. Numerical experiments indicate that our error estimate is an optimal one.

Before we present the main result, some preliminary work has to be done.

Lemma 4.1. *Let δ_{ik} be the Kronecker delta. Then, for $k = 1, 2, \ldots, n$, the solution of the linear system*

$$\begin{cases} \displaystyle\sum_{j=0}^{n} \frac{\xi_j^{(k)}}{\hat{x}_i - x_j} = \frac{\delta_{ik}}{h}, \quad i = 1, 2, \ldots, n, \\ \displaystyle\sum_{j=0}^{n} \xi_j^{(k)} = 0 \end{cases} \tag{4.3}$$

has the following expression

$$\xi_j^{(k)} = -\frac{h(x_k - x_0) l_{nj}(\hat{x}_{j+1}) l_{nk}(\hat{x}_k)}{4(\hat{x}_{j+1} - x_k)(\hat{x}_{j+1} - x_0)}, \quad j = 0, 1, 2, \ldots, n, \tag{4.4}$$

where $\hat{x}_{n+1} = x_n + h/2$ and

$$l_{ni}(x) = \frac{\prod_{j=0, j\neq i}^{n} (x - x_j)}{\prod_{j=0, j\neq i}^{n} (x_i - x_j)}. \tag{4.5}$$

Proof. For simplicity of exposition, we introduce the notation

$$\Delta_{n+1}(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n; x_0, x_1, \ldots, x_n) = \begin{vmatrix} \frac{1}{\hat{x}_1 - x_0} & \frac{1}{\hat{x}_1 - x_1} & \cdots & \frac{1}{\hat{x}_1 - x_n} \\ \frac{1}{\hat{x}_2 - x_0} & \frac{1}{\hat{x}_2 - x_1} & \cdots & \frac{1}{\hat{x}_2 - x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{\hat{x}_n - x_0} & \frac{1}{\hat{x}_n - x_1} & \cdots & \frac{1}{\hat{x}_n - x_n} \\ 1 & 1 & \cdots & 1 \end{vmatrix}. \tag{4.6}$$

Subtracting the last column of $\Delta_{n+1}(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n; x_0, x_1, \ldots, x_n)$ from all the rest columns, taking out the common factors in the rows and columns and, in the resulting determinant, subtracting the n-th row from all the preceding rows, taking out the common factors in the rows and columns and expanding in the last column, we get

$$\begin{aligned} &\Delta_{n+1}(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n; x_0, x_1, \ldots, x_n) \\ &\quad = \prod_{i=1}^{n-1} (\hat{x}_n - \hat{x}_i) \prod_{i=0}^{n-1} \frac{(x_i - x_n)}{(\hat{x}_{i+1} - x_n)(\hat{x}_n - x_i)} \Delta_n(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_{n-1}; x_0, x_1, \ldots, x_{n-1}). \end{aligned} \tag{4.7}$$

Starting from this recurrence relation, we arrive at

$$\begin{aligned} &\Delta_{n+1}(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n; x_0, x_1, \ldots, x_n) \\ &\quad = \prod_{1\le i<l\le n} (\hat{x}_l - \hat{x}_i) \prod_{0\le i<l\le n} (x_i - x_l) \prod_{1\le i\le l\le n} \frac{1}{(\hat{x}_i - x_l)} \prod_{0\le i<l\le n} \frac{1}{(\hat{x}_l - x_i)}. \end{aligned} \tag{4.8}$$

Using the same derivation, we obtain

$$\begin{aligned}\Delta_n^{k,j} &:= \Delta_n(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_{k-1}, \hat{x}_{k+1}, \cdots, \hat{x}_n; x_0, x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) \\ &= \prod_{1\le i<l\le n, i,l\ne k} (\hat{x}_l - \hat{x}_i) \prod_{0\le i<l\le n, i,l\ne j} (x_i - x_l) \prod_{1\le i\le l\le n, i\ne k, l\ne j} \frac{1}{(\hat{x}_i - x_l)} \prod_{0\le i<l\le n, l\ne k, i\ne j} \frac{1}{(\hat{x}_l - x_i)}.\end{aligned} \tag{4.9}$$

Since, for $k = 1, 2, \dots, n$, $j = 0, 1, \dots, n$,

$$\frac{\prod\limits_{1\le i<l\le n, i,l\ne k} (\hat{x}_l - \hat{x}_i)}{\prod\limits_{1\le i<l\le n} (\hat{x}_l - \hat{x}_i)} = \frac{1}{(-1)^{n-k} \prod\limits_{1\le i\le n, i\ne k} (\hat{x}_k - \hat{x}_i)},$$

$$\frac{\prod\limits_{0\le i<l\le n, i,l\ne j} (x_i - x_l)}{\prod\limits_{0\le i<l\le n} (x_i - x_l)} = \frac{1}{(-1)^{j} \prod\limits_{0\le i\le n, i\ne j} (x_j - x_i)},$$

$$\frac{\prod\limits_{1\le i\le l\le n} (\hat{x}_i - x_l)}{\prod\limits_{1\le i\le l\le n, i\ne k, l\ne j} (\hat{x}_i - x_l)} = \prod_{k\le i\le n} (\hat{x}_k - x_i) \prod_{1\le i\le j, i\ne k} (\hat{x}_i - x_j),$$

$$\frac{\prod\limits_{0\le i<l\le n} (\hat{x}_l - x_i)}{\prod\limits_{0\le i<l\le n, l\ne k, i\ne j} (\hat{x}_l - x_i)} = \prod_{0\le i<k} (\hat{x}_k - x_i) \prod_{j<i\le n, i\ne k} (\hat{x}_i - x_j),$$

we have, by Cramer's rule,

$$\begin{aligned}\xi_j^{(k)} &= \frac{(-1)^{k+j+1}}{h} \frac{\Delta_n^{k,j}}{\Delta_{n+1}(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n; x_0, x_1, \dots, x_n)} \\ &= \frac{(-1)^{k+j+1}}{h} \frac{\prod\limits_{0\le i\le n} (\hat{x}_k - x_i) \prod\limits_{1\le i\le n, i\ne k} (\hat{x}_i - x_j)}{(-1)^{n-k+j} \prod\limits_{1\le i\le n, i\ne k} (\hat{x}_k - \hat{x}_i) \prod\limits_{0\le i\le n, i\ne j} (x_j - x_i)} \\ &= \frac{\prod\limits_{0\le i\le n} (\hat{x}_k - x_i) \prod\limits_{1\le i\le n, i\ne k} (\hat{x}_{j+1} - x_i)}{h \prod\limits_{1\le i\le n, i\ne k} (x_k - x_i) \prod\limits_{0\le i\le n, i\ne j} (x_j - x_i)} \\ &= \frac{(x_k - x_0)(\hat{x}_k - x_k)(\hat{x}_{j+1} - x_j) \prod\limits_{0\le i\le n, i\ne k} (\hat{x}_k - x_i) \prod\limits_{0\le i\le n, i\ne j} (\hat{x}_{j+1} - x_i)}{h(\hat{x}_{j+1} - x_0)(\hat{x}_{j+1} - x_k) \prod\limits_{0\le i\le n, i\ne k} (x_k - x_i) \prod\limits_{0\le i\le n, i\ne j} (x_j - x_i)} \\ &= -\frac{h(x_k - x_0) l_{nk}(\hat{x}_k) l_{nj}(\hat{x}_{j+1})}{4(\hat{x}_{j+1} - x_0)(\hat{x}_{j+1} - x_k)},\end{aligned} \tag{4.10}$$

where we have used

$$\hat{x}_k - \hat{x}_i = x_k - x_i, \qquad \hat{x}_i - x_j = -(\hat{x}_{j+1} - x_i).$$

The proof is then complete. □

One can see that $l_{ni}(x)$, defined by (4.5), is none other than the Lagrange interpolation basis function that satisfies

$$l_{ni}(x_j) = \delta_{ij}.$$

Moreover, we have the result below.

Lemma 4.2. *For the function $l_{ni}(x)$ defined by (4.5),*

$$0 < l_{ni}(\hat{x}_i) \le C\sqrt{\frac{n-i+1}{i}}, \quad 1 \le i \le n \tag{4.11}$$

and

$$0 < l_{ni}(\hat{x}_{i+1}) \le C\sqrt{\frac{i+1}{n-i}}, \quad 0 \le i \le n-1. \tag{4.12}$$

Proof. By definition, we have

$$l_{ni}(\hat{x}_i) = \frac{\prod_{j=0,j\neq i}^{n} (\hat{x}_i - x_j)}{\prod_{j=0,j\neq i}^{n} (x_i - x_j)} = \prod_{j=1}^{i}\left(1-\frac{1}{2j}\right)\prod_{j=1}^{n-i}\left(1+\frac{1}{2j}\right)$$

for $1 \le i < n$ and

$$l_{ni}(\hat{x}_i) = \prod_{j=1}^{i}\left(1-\frac{1}{2j}\right)$$

for $i = n$, by which the positiveness of $l_{ni}(\hat{x}_i)$ is obtained. Now, by using the formula [27],

$$\prod_{j=1}^{n}\left(1+\frac{\beta}{n}\right) = \frac{n^{\beta}}{\Gamma(1+\beta)} + O(n^{\beta-1}), \tag{4.13}$$

we get the upper bound of $l_{ni}(\hat{x}_i)$ in (4.11). In an analogous way, we can prove (4.12), which concludes the proof. □

Lemma 4.3. *Let $\mathcal{A}_n = (a_{ij})_{n\times n}$ be the matrix defined in (4.2). Then*

(i) *$\mathcal{A}_n$ is a symmetric Toeplitz matrix. Moreover, $\mathcal{A}_n$ is strictly diagonally dominant and as a consequent, the linear system (4.1) has a unique solution;*
(ii) *$-\mathcal{A}_n$ is an M-matrix;*
(iii) *$\mathcal{A}_n$ is a persymmetric matrix, i.e.,*

$$\mathcal{A}_n = E_n \mathcal{A}_n^{\mathrm{T}} E_n, \tag{4.14}$$

where

$$E_n = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Proof. Obviously, $\mathcal{A}_n$ is a symmetric Toeplitz matrix. By (4.2),

$$a_{ii} = -\frac{4}{h} < 0 \tag{4.15}$$

and for $i \neq j$,

$$a_{ij} = \frac{4}{[4(j-i)^2 - 1]h} > 0. \tag{4.16}$$

Moreover,

$$\sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n}\left(\frac{1}{x_{j-1}-\hat{x}_i} - \frac{1}{x_j - \hat{x}_i}\right) = \frac{b-a}{(b-\hat{x}_i)(a-\hat{x}_i)} < 0. \tag{4.17}$$

Thus $\mathcal{A}_n$ is a strictly diagonally dominant matrix. The result that $-\mathcal{A}_n$ is an M-matrix follows from (4.15)–(4.17). (4.14) can be directly verified by straightforward calculation and by noting $\mathcal{A}_n$ is a symmetric Toeplitz matrix. The proof is complete. □

Lemma 4.4. *Let $\mathcal{A}_n^{-1} = (b_{ik})_{n\times n}$ be the inverse matrix of $\mathcal{A}_n$, defined in (4.2). Then,*

$$|b_{ik}| \le Ch \min\{\ln(k+1), \ln(n-k+2)\}. \tag{4.18}$$

Proof. We rewrite the linear system (4.1) by

$$\begin{cases} \sum_{j=0}^{n} -\frac{u_{j+1}-u_j}{h}\frac{1}{\hat{x}_i - x_j} = \frac{1}{h} f(\hat{x}_i), \quad i = 1, 2, \ldots, n, \\ \sum_{j=0}^{n} -\frac{u_{j+1}-u_j}{h} = 0, \end{cases}$$

where $u_{n+1} = u_0 = 0$. By Lemma 4.1, the solution of this linear system can be expressed as

$$-\frac{u_{j+1}-u_j}{h} = -\sum_{k=1}^{n} f(\hat{x}_k)\frac{h(x_k-x_0)l_{nj}(\hat{x}_{j+1})l_{nk}(\hat{x}_k)}{4(\hat{x}_{j+1}-x_k)(\hat{x}_{j+1}-x_0)}, \quad j = 0, 1, 2, \ldots, n.$$

As a result, by using

$$u_i = \sum_{j=0}^{i-1} \frac{u_{j+1}-u_j}{h} \cdot h,$$

we get

$$u_i = \sum_{j=0}^{i-1}\sum_{k=1}^{n} f(\hat{x}_k)\frac{h^2(x_k-x_0)l_{nj}(\hat{x}_{j+1})l_{nk}(\hat{x}_k)}{4(\hat{x}_{j+1}-x_k)(\hat{x}_{j+1}-x_0)}, \quad i = 1, 2, \ldots, n. \tag{4.19}$$

It follows that

$$b_{ik} = \sum_{j=0}^{i-1} \frac{h^2(x_k-x_0)l_{nj}(\hat{x}_{j+1})l_{nk}(\hat{x}_k)}{4(\hat{x}_{j+1}-x_k)(\hat{x}_{j+1}-x_0)},$$

which can be simplified as

$$b_{ik} = h\sum_{j=0}^{i-1} \frac{kl_{nj}(\hat{x}_{j+1})l_{nk}(\hat{x}_k)}{(2j+1-2k)(2j+1)}. \tag{4.20}$$

Armed with this formula, we are ready to estimate b_{ik}. First consider the case where $i \le k$. We have, by Lemma 4.2,

$$|b_{ik}| = -b_{ik} = h\sum_{j=0}^{i-1} \frac{kl_{nj}(\hat{x}_{j+1})l_{nk}(\hat{x}_k)}{(2k-2j-1)(2j+1)} \tag{4.21}$$

and

$$\begin{aligned} |b_{ik}| &\le Ch\sum_{j=0}^{i-1}\left\{\frac{k}{(2k-2j-1)(2j+1)}\sqrt{\frac{j+1}{n-j}}\sqrt{\frac{n-k+1}{k}}\right\} \\ &\le Ch\sum_{j=0}^{i-1}\frac{k}{(2k-2j-1)(2j+1)} \\ &\le Ch\sum_{j=0}^{k-1}\left(\frac{1}{2k-2j-1}+\frac{1}{2j+1}\right) \\ &\le Ch\left(1+\frac{1}{3}+\cdots+\frac{1}{2k-1}\right). \end{aligned} \tag{4.22}$$

Using the estimate [28]

$$\frac{1}{2(k+1)} < \sum_{i=1}^{k}\frac{1}{i} - \ln k - \gamma < \frac{1}{2k}, \tag{4.23}$$

where γ is the Euler constant, we get

$$|b_{ik}| \le Ch\ln(k+1), \quad i \le k. \tag{4.24}$$

As for $i > k$, by an argument similar to that of (4.22), we have

$$\begin{aligned} -b_{ik} &= h\sum_{j=0}^{k-1}\frac{kl_{nj}(\hat{x}_{j+1})l_{nk}(\hat{x}_k)}{(2k-2j-1)(2j+1)} - h\sum_{j=k}^{i-1}\frac{kl_{nj}(\hat{x}_{j+1})l_{nk}(\hat{x}_k)}{(2j+1-2k)(2j+1)} \\ &\le h\sum_{j=0}^{k-1}\frac{kl_{nj}(\hat{x}_{j+1})l_{nk}(\hat{x}_k)}{(2k-2j-1)(2j+1)} \\ &\le Ch\ln(k+1). \end{aligned} \tag{4.25}$$

Note that $-\mathcal{A}_n$ is an M-matrix. Since a nonsingular M-matrix is inverse-positive (cf. [29]), that is, $-\mathcal{A}_n^{-1} \geq 0$, which implies $b_{ik} \leq 0$ for any i, k. Hence, it follows from (4.25) that

$$|b_{ik}| = -b_{ik} \leq Ch \ln(k+1), \quad i > k. \tag{4.26}$$

Combining (4.24) with (4.26) yields

$$|b_{ik}| \leq Ch \ln(k+1), \quad \text{for any } i, k. \tag{4.27}$$

By Lemma 4.3(iii), we get

$$\mathcal{A}_n^{-1} = E_n (\mathcal{A}_n^{-1})^{\mathrm{T}} E_n,$$

which implies

$$b_{ik} = b_{n+1-k,n+1-i}. \tag{4.28}$$

Note that $\mathcal{A}_n^{-1}$ is symmetric. It comes from (4.27) and (4.28) that

$$|b_{ik}| = |b_{n+1-i,n+1-k}| \leq Ch \ln(n-k+2). \tag{4.29}$$

Incorporating this estimate with (4.27) leads to (4.18). □

Now we are ready to present our main result of this section.

Theorem 4.5. *Assume that $u(x)$, the solution of the finite-part integral equation (1.1), belongs to $C^{2+\alpha}[a, b](0 < \alpha < 1)$. Then, for the solution of linear system (4.1), there holds the error estimate*

$$\max_{1 \leq i \leq n} |u(\hat{x}_i) - u_i| \leq Ch. \tag{4.30}$$

Proof. Let $\mathbf{U}_e = (u(\hat{x}_1), u(\hat{x}_2), \ldots, u(\hat{x}_n))^{\mathrm{T}}$ be the exact solution vector. Then, from (4.1), we see that

$$\mathbf{U}_e - \mathbf{U}_a = \mathcal{A}_n^{-1}(\mathcal{A}_n \mathbf{U}_e - \mathbf{F}_e), \tag{4.31}$$

which implies

$$u(\hat{x}_i) - u_i = \sum_{k=1}^{n} b_{ik} \mathcal{E}_{n,0}(u; \hat{x}_k), \quad i = 1, 2, \ldots, n, \tag{4.32}$$

where $\{b_{ik}\}$ are the entries of $\mathcal{A}_n^{-1}$ and $\mathcal{E}_{n,0}(u; \hat{x}_k)$ is defined in (2.2). By Theorem 3.2 and Lemma 4.4, we obtain

$$\begin{aligned}
|u(\hat{x}_i) - u_i| &\leq Ch^{1+\alpha} \sum_{k=1}^{n} [1 + \eta^2(\hat{x}_k) h^{1-\alpha}] |b_{ik}| \\
&\leq Ch^{2+\alpha} \sum_{k=1}^{n} \min\{\ln(k+1), \ln(n-k+2)\} \\
&\quad + Ch \sum_{k=1}^{n} \left[\frac{h^2}{(\hat{x}_k - a)^2} + \frac{h^2}{(b - \hat{x}_k)^2} \right] \min\{\ln(k+1), \ln(n-k+2)\} \\
&\leq Ch^{1+\alpha} |\ln h| + Ch \sum_{k=1}^{n} \left\{ \frac{\ln(k+1)}{(2k-1)^2} + \frac{\ln(n-k+2)}{[2(n-k)+1]^2} \right\} \\
&\leq Ch. \quad \square
\end{aligned} \tag{4.33}$$

Remark. By Theorem 3.2, the truncation error of the collocation equation (4.1), i.e., $\mathcal{E}_{n,0}(u; \hat{x}_k)$, will not vanishes if the collocation point $\hat{x}_k$ is very close to the endpoints a and b, since in this case $\eta(\hat{x}_k) = O(h^{-1})$. However, we manager to obtain the one-order nodal error estimate (4.30). The superconvergence error estimate of the midpoint rule and the delicate estimate of the entries of $\mathcal{A}_n^{-1}$ play key roles in the argument.

Table 1
Errors for the case where $s = x_{[n/4]} + (1+\tau)h/2$.

n	$\widetilde{Q}_{n,0}(u)$			$Q_{n,0}(u)$		
	$\tau = -1/3$	$\tau = 0$	$\tau = 1/3$	$\tau = -1/3$	$\tau = 0$	$\tau = 1/3$
31	9.5426E−4	4.9886E−4	4.3517E−5	3.0546E−1	4.9886E−4	3.3277E−1
63	2.3609E−4	1.2381E−4	1.1535E−5	3.2257E−1	1.2381E−4	3.3649E−1
127	5.8706E−5	3.0833E−5	2.9597E−6	3.3128E−1	3.0833E−5	3.3830E−1
255	1.4636E−5	7.6928E−6	7.4907E−7	3.3567E−1	7.6928E−6	3.3920E−1
511	3.6541E−6	1.9212E−6	1.8839E−7	3.3788E−1	1.9212E−6	3.3964E−1
1023	9.1289E−7	4.8006E−7	4.7236E−8	3.3898E−1	4.8006E−7	3.3987E−1
Error order	$h^{1.988}$	$h^{1.987}$	$h^{1.953}$	$h^{-0.030}$	$h^{1.987}$	$h^{-0.006}$

Table 2
Errors for the case where $s = x_{n-1} + (1+\tau)h/2$.

n	$\widetilde{Q}_{n,0}(u)$			$Q_{n,0}(u)$		
	$\tau = -1/3$	$\tau = 0$	$\tau = 1/3$	$\tau = -1/3$	$\tau = 0$	$\tau = 1/3$
31	5.0507E−1	8.3440E−1	1.6392E0	5.7149E0	8.3440E−1	3.6858E0
63	4.9623E−1	8.2233E−1	1.6193E0	5.8231E0	8.2233E−1	3.7647E0
127	4.9215E−1	8.1660E−1	1.6096E0	5.8766E0	8.1660E−1	3.8032E0
255	4.9020E−1	8.1381E−1	1.6049E0	5.9032E0	8.1381E−1	3.8223E0
511	4.8924E−1	8.1244E−1	1.6026E0	5.9164E0	8.1244E−1	3.8317E0
1023	4.8877E−1	8.1176E−1	1.6014E0	5.9231E0	8.1176E−1	3.8364E0
Error order	$h^{0.009}$	$h^{0.008}$	$h^{0.007}$	$h^{-0.010}$	$h^{0.008}$	$h^{-0.011}$

5. Numerical examples

In this section, we present some numerical examples to confirm our theoretical analysis given in the above sections.

Example 5.1. Consider the finite-part integral in the left hand side of (1.1) with a smooth integral function $u(x) = x^3$ and $a = 0$, $b = 1$. The exact value of this finite-part integral is

$$\frac{3}{2} + 3s + \frac{1}{s-1} + 3s^2 \ln \frac{1-s}{s}.$$

Both the midpoint rule (2.2) and the modified midpoint rule (3.1) are used. Numerical results are presented in Table 1 where the singular point s is chosen to be a dynamic one, namely, $x_{[n/4]} + (1+\tau)h/2$. Numerical estimates of the error order are presented in the last row, which is obtained from a least square fit. One can see from the left column that the error bound for the modified midpoint rule is $O(h^2)$, independent of the values of τ. However, from the right column, we see that if s is not a superconvergence point ($\tau \neq 0$), the midpoint rule is divergent, while the error bound is $O(h^2)$ if s is a superconvergence point ($\tau = 0$). In Table 2 we present the results for singular point $s = x_{n-1} + (1+\tau)h/2$. In this case, $\eta(s) = O(h^{-1})$. One can see that both the midpoint rule and the modified midpoint rule are divergent. These results agree quite well with the estimates in Theorems 3.1 and 3.2.

Example 5.2. Now we consider an example with different regularities. Let $a = -b = -1$, $s = 0$ and

$$u(x) = x^2 + x + [2 + \mathrm{sign}(x)]|x|^{p+\alpha}, \quad p = 1, 2, 3.$$

Obviously, $u(x) \in C^{p+\alpha}[-1, 1]$. The exact value of the finite-part integral is

$$2 + \frac{4}{\alpha + p - 1}.$$

First we adopt a uniform mesh in which the local coordinate of s is $\tau = 0$. In this case, the modified midpoint rule coincides with the midpoint rule. The numerical results are presented in the right half of Table 3. We can see that if the integrand function $u(x)$ is smooth enough ($p = 3$), the error bound will be $O(h^2)$, and if $u(x)$ has less regularity ($p = 1, 2$), the error bound will be approximately $O(h^{p-1+\alpha})$, dependent of the regularity parameters p and α, which confirms that the estimates given in Theorems 3.1 and 3.2 and (3.13) are optimal. Secondly, we translate the interior nodes of the uniform mesh so that the local coordinate of s is $\tau = -1/3$. The numerical results for this quasi-uniform mesh are given in the left half of Table 3, which confirms again the optimization of the estimates given in Theorem 3.1 and (3.13).

Example 5.3. Now, we consider an example of solving the finite-part integral Eq. (1.1) by collocation scheme (4.1). Let $a = 0$, $b = 1$ and

$$f(s) = \frac{6}{5} + \frac{3}{2}s + 2s^2 + 3s^3 + 6s^4 + \frac{1}{s-1} + 6s^5 \ln \frac{1-s}{s}.$$

Table 3
Errors for the case of different regularities ($\alpha = 1/2$).

n	$\widetilde{\mathcal{Q}}_{n,0}(u)(\tau = -1/3)$			$\mathcal{Q}_{n,0}(u)(\tau = 0)$		
	$p = 3$	$p = 2$	$p = 1$	$p = 3$	$p = 2$	$p = 1$
255	1.5347E−4	1.0082E−3	4.9298E−1	7.0683E−5	1.3328E−3	3.6778E−1
511	3.8807E−5	3.6503E−4	3.4824E−1	1.8056E−5	4.8626E−4	2.5978E−1
1023	9.7860E−6	1.3127E−4	2.4612E−1	4.5850E−6	1.7576E−4	1.8359E−1
2047	2.4623E−6	4.6975E−5	1.7399E−1	1.1592E−6	6.3119E−5	1.2979E−1
4095	6.1846E−7	1.6752E−5	1.2302E−1	2.9216E−7	2.2563E−5	9.1761E−2
8191	1.5514E−7	5.9589E−6	8.6980E−2	7.3462E−8	8.0396E−6	6.4881E−2
Error order	$h^{1.988}$	$h^{1.479}$	$h^{0.500}$	$h^{1.980}$	$h^{1.473}$	$h^{0.500}$

Table 4
Errors for the solution of the finite-part integral equation.

n	trunc-e_∞	e_∞
32	1.7408E0	1.9082E−2
64	1.6797E0	9.2449E−3
128	1.6502E0	4.5444E−3
256	1.6360E0	2.2522E−3
512	1.6290E0	1.1210E−3
Error order	$h^{0.024}$	$h^{1.022}$

The exact solution is $u(x) = x^6$. We examine the maximal nodal error and the maximal truncation error, defined respectively by

$$e_\infty = \max_{1\le i\le n} |u(\hat{x}_i) - u_i|, \qquad \text{trunc-}e_\infty = \max_{1\le i\le n} |\mathcal{E}_{n,0}(u; \hat{x}_i)|, \tag{5.1}$$

where u_i $(i = 1, 2, \ldots, n)$ denotes the approximation of $u(x)$ at $\hat{x}_i$ and $\mathcal{E}_{n,0}(u; \hat{x}_i)$ is defined in (2.2). Numerical results presented in Table 4 indicate that, although the maximal truncation error is $O(1)$, the maximal nodal error can still reach $O(h)$, the same order as in Theorem 4.5.

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