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JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

Journal of Computational and Applied Mathematics 223 (2009) 598–613

www.elsevier.com/locate/cam

# Superconvergence of the composite Simpson's rule for a certain finite-part integral and its applications

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Received 6 September 2007; received in revised form 10 January 2008

## Abstract

The superconvergence phenomenon of the composite Simpson's rule for the finite-part integral with a third-order singularity is studied. The superconvergence points are located and the superconvergence estimate is obtained. Some applications of the superconvergence result, including the evaluation of the finite-part integrals and the solution of a certain finite-part integral equation, are also discussed and two algorithms are suggested. Numerical experiments are presented to confirm the superconvergence analysis and to show the efficiency of the algorithms.



*MSC:* 65D32; 65D30; 65R20

*Keywords:* Finite-part integral; Finite-part integral equation; Composite Simpson's rule; Superconvergence

## 1. Introduction

Consider the following integral:

$$I_p(a,b;s,f) := \,=\!\!\!\!\!\!\int_a^b \frac{f(t)}{(t-s)^{p+1}}\mathrm{d}t, \quad s \in (a,b),\ p = 1, 2, \tag{1.1}$$

where $=\!\!\!\!\!\!\int$ denotes a Hadamard finite-part integral (or hypersingular integral) with $p+1$-order singularity and $s$ is the singular point. Integrals of this type are often encountered in many physical problems, such as in fracture mechanics, elasticity problems, acoustics as well as electromagnetic scattering (see, e.g., [2,5,6,11,15,18,32]).

Hypersingular integral (1.1) must be understood in the Hadamard finite-part sense. There are several definitions, equivalent mathematically, for this finite-part integral in the literature (e.g., see, [10,13,16,21–23]). Here, we use the

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0377-0427/$ - see front matter 
doi:10.1016/j.cam.2008.02.001

definition:

$$\oint_a^b \frac{f(t)}{(t-s)^{p+1}}\mathrm{d}t = \lim_{\varepsilon\to 0}\left\{\int_a^{s-\varepsilon}\frac{f(t)}{(t-s)^{p+1}}\mathrm{d}t + \int_{s+\varepsilon}^b \frac{f(t)}{(t-s)^{p+1}}\mathrm{d}t - \frac{2f^{(p-1)}(s)}{\varepsilon}\right\}, \quad s\in(a,b),$$
$$p = 1, 2, \tag{1.2}$$

$f(t)$ is said to be finite-part integrable (or integrable in the Hadamard sense) with the weight $(t-s)^{-p-1}$ if the limit on the right-hand side of (1.2) exists. By this definition and through direct calculation, we have

$$\oint_a^b \frac{(t-s)^{p+l}}{(t-s)^{p+1}}\mathrm{d}t = \begin{cases} \ln\dfrac{b-s}{s-a}, & l = 0,\\ \dfrac{1}{l}[(b-s)^l - (a-s)^l], & l \neq 0. \end{cases} \tag{1.3}$$

In some special cases, say, $f(t)$ is a polynomial in $t$, (1.1) can be evaluated analytically by first expanding $f(t)$ into Taylor series at $s$ and then using (1.3). This technique will be frequently used throughout this paper. Generally speaking, finite-part integrals usually cannot be evaluated directly and numerical methods have to be employed.

Due to the hypersingularity of the kernel, many quadrature rules for the finite-part integrals, such as the trapezoidal rule and Simpson's rule, are less accurate than their counterparts for Riemann integrals. These rules are even invalid in some cases, for example, the trapezoidal rule is invalid for (1.1) with $p > 1$. How to evaluate the finite-part integrals efficiently is an interesting problem that has drawn many authors' attention. In the recent decays, there have been a lot of works in developing efficient quadrature methods, which include the Gaussian quadrature method [1,8,16,20,22,24,26,28], the composite Newton–Cotes method [9,21,31,33–35], the sigmoidal transformation method [7,10] and some other methods [14,17].

The Gaussian quadrature and the sigmoidal transformation method are efficient when the integrand function $f(t)$ is smooth enough. A main disadvantage is the restriction on mesh selection. For example, in the Gaussian quadrature method, the integral is approximated in terms of $f(t)$ at some special points, such as Gaussian points. In some physical problems, one needs to solve a hypersingular integral equation coupled with some domain equations where $f(t)$ is usually unknown and less smooth. The discretization of the integral equation should be made on a mesh which well fits the approximation to the domain equations, such as finite element approximation or finite difference approximation. In this case the composite Newton–Cotes method becomes a competitive one. In comparison with Gaussian quadrature method or the sigmoidal transformation method, the Newton–Cotes method is much easier for implementation and has relatively less restriction on the selection of mesh points. The Newton–Cotes method was first introduced in [21] for evaluating the hypersingular integral (1.1) with $p = 1$ and later in [9] for evaluating (1.1) with $p = 2$.

It has been shown in [21] that the composite Newton–Cotes method may be invalid when the singular point is close to a mesh point. To assure a good accuracy, one must choose the mesh very carefully such that the singular point is located near the center of a certain subinterval. This is the so-called mesh selection problem of the composite Newton–Cotes method. A significant improvement was made in [27] where some new error estimates were obtained for the classical Newton–Cotes formulae, which largely extends the applicability of the method and makes the mesh selection problem less serious. Besides, some indirect methods were suggested in [9,31] to remove the mesh selection problem.

In the following discussion, we shall confine ourselves to the evaluation of finite-part integral (1.1) with $p = 2$. Subdivide the interval $[a, b]$ into $n$ subintervals using the mesh points $a = t_0 < t_1 < \cdots < t_n = b$ and denote $h_i = t_i - t_{i-1}$. Let $t_{i-1/2}$ be the midpoint of subinterval $[t_{i-1}, t_i]$ $(i = 1, 2, \ldots, n)$. Then the integral function $f(t)$ can be approximated by its piecewise quadratic polynomial interpolant

$$f_Q(t) = \sum_{i=0}^{2n}\phi_i(t) f(t_{\frac{i}{2}}),$$

where $\phi_i(t) (i = 0, 1, \ldots, 2n)$ are the basis functions, given by

$$\phi_{2i}(t) = \begin{cases} \dfrac{2(1-\delta_0^i)}{h_i^2}(t-t_{i-1})(t-t_{i-\frac{1}{2}}), & t \in [t_{i-1}, t_i], \\ \dfrac{2(1-\delta_n^i)}{h_{i+1}^2}(t-t_{i+\frac{1}{2}})(t-t_{i+1}), & t \in [t_i, t_{i+1}], \\ 0, & \text{otherwise}, \end{cases} \tag{1.4}$$

$$\phi_{2j+1}(t) = \begin{cases} -\dfrac{4}{h_{j+1}^2}(t-t_j)(t-t_{j+1}), & t \in [t_j, t_{j+1}], \\ 0, & \text{otherwise}, \end{cases} \tag{1.5}$$

with $i = 0, 1, \ldots, n$ and $j = 0, 1, \ldots, n-1$. Replacing $f(t)$ in (1.1) by $f_Q(t)$, we obtain the composite Simpson's rule:

$$I_2(a, b; s, f_Q) = \not\int_a^b \frac{f_Q(t)}{(t-s)^3}\mathrm{d}t = \sum_{i=0}^{2n} \omega_i(s) f(t_{\frac{i}{2}}), \tag{1.6}$$

where $\omega_i(s)$ are the Cotes coefficients. If $s \neq t_j$ for any $j = 0, 1, \ldots, n$, we have

$$\begin{aligned} \omega_{2i}(s) \;=\; & \frac{1-\delta_0^i}{h_i^2}\left\{\left[2s-(t_{i-1}+t_{i-\frac{1}{2}})\right]\left(\frac{1}{s-t_i}-\frac{1}{s-t_{i-1}}\right)+2\ln\left|\frac{t_i-s}{t_{i-1}-s}\right|\right\} \\ & + \frac{1-\delta_n^i}{h_{i+1}^2}\left\{[2s-(t_{i+\frac{1}{2}}+t_{i+1})]\left(\frac{1}{s-t_{i+1}}-\frac{1}{s-t_i}\right)+2\ln\left|\frac{t_{i+1}-s}{t_i-s}\right|\right\} \\ & + \frac{\delta_0^i}{2(t_0-s)^2}-\frac{\delta_n^i}{2(t_n-s)^2}, \quad i = 0, 1, \ldots, n; \\ \omega_{2j+1}(s) = & -\frac{2}{h_{j+1}^2}\left\{[2s-(t_j+t_{j+1})]\left(\frac{1}{s-t_{j+1}}-\frac{1}{s-t_j}\right)+2\ln\left|\frac{t_{j+1}-s}{t_j-s}\right|\right\}, \quad j = 0, 1, \ldots, n-1. \end{aligned}$$

The accuracy of the above Simpson's rule is proved to be $O(h)$ in general (cf. [9]). We find that the accuracy of this rule can be one order higher when the singular point $s$ coincides with some special points. We refer to this phenomenon as superconvergence. Superconvergence phenomenon of this kind has been investigated in [30] for the trapezoidal rule of (1.1) with $p = 1, 2$ and in [29] for Simpson's rule of (1.1) with $p = 1$. However, up until now, the superconvergence of Simpson's rule for the finite-part integrals with a third-order singularity has not been investigated. Here we shall locate the superconvergence points and obtain the superconvergence estimate for the composite Simpson's rule (1.6). More importance is that we shall discuss several applications of the superconvergence result, including the evaluation of finite-part integral (1.1) with $p = 2$ in the general case ($s$ is not a superconvergence point) and the solution of a certain finite-part integral equation with a third-order singularity.

The rest of this paper is organized as follows. In the next section, the main result of this paper is introduced and the related theoretical analysis is given. In Section 3, some applications of the superconvergence result are discussed and two algorithms are proposed. In the last section, some numerical results are presented to validate our analysis and to show the high efficiency of our algorithms.

## 2. Superconvergence result

In this section, we study the superconvergence of the composite Simpson's rule for the Hadamard finite-part integral with a third-order singularity. We first present our main result of the superconvergence in Theorem 2.1 and then give the proof.

### 2.1. The main result

**Theorem 2.1.** *Let $I_2(a, b; s, f_Q)$ be computed by (1.6) with a uniform mesh and $s = t_m + h/2 \pm h/3$. Then there exists a positive constant C, independent of h and s, such that*

$$|I_2(a,b;s,f) - I_2(a,b;s,f_Q)| \le \begin{cases} C[1+\eta(s)h^{2-\alpha}]h^{1+\alpha}, & f(t) \in C^{3+\alpha}[a,b], \\ C[|\ln h| + \eta(s)h]h^2, & f(t) \in C^4[a,b], \\ C[1+\eta(s)h]h^2, & f(t) \in C^{4+\alpha}[a,b], \end{cases} \tag{2.1}$$

*where* $0 < \alpha < 1$ *and*

$$\eta(s) = \max\left\{\frac{1}{(s-a)^2}, \frac{1}{(b-s)^2}\right\}. \tag{2.2}$$

Theorem 2.1 states that for the superconvergence points belonging to any closed subinterval of $(a, b)$ away from the endpoints, the convergence rate of Simpson's rule is $O(h^2)$, i.e., the same as that we have when we use the trapezoidal rule (see [30]). However, for those points near the endpoints $a$ and $b$, the rate of convergence given by Theorem 2.1 is $O(h)$, which is one order higher than that of the trapezoidal rule. Furthermore, both rules exhibit different natures when we use them and their superconvergence results to solve finite-part integral equations. The rate of convergence by using the composite Simpson's rule will be one order higher than that by using composite trapezoidal rule, which will be illustrated by numerical experiments in Section 4.

### 2.2. Preliminaries for the proof

In the following analysis, $C$ will denote a generic positive constant independent of $h$ and $s$ with possibly different values in different places. Let $m$ be a positive integer, $\tau$ be a real number and

$$\mathcal{H}_m^i(\tau) := 1 + (m+1-i+\tau)\ln\left|\frac{2m+1-2i+2\tau}{2m+3-2i+2\tau}\right|, \tag{2.3}$$

$$\mathcal{M}_m^i(\tau) := 3\mathcal{H}_m^i(\tau) + \frac{1}{4(m+1-i+\tau)^2 - 1}, \tag{2.4}$$

where $i$ is a positive integer, satisfying $1 \le i \le 2m+1$.

**Lemma 2.2** (*[30], Lemma 3.1*). *For* $|\tau| < 1/2$, *there hold the following estimates*

$$\text{(i)} \sum_{i=1}^{2m+1} |m+1-i+\tau|^\mu |\mathcal{H}_m^i(\tau)| \le C|\ln(1-4\tau^2)| + \sum_{i=1}^{m} \frac{1}{i^{2-\mu}}, \quad 0 \le \mu < 2; \tag{2.5}$$

$$\text{(ii)} \left|\sum_{i=1}^{2m+1} (m+1-i+\tau)\mathcal{H}_m^i(\tau)\right| \le C. \tag{2.6}$$

**Lemma 2.3.** *For* $|\tau| < 1/2$, *there hold*

$$\text{(i)} \sum_{i=1}^{2m+1} |m+1-i+\tau|^\mu |\mathcal{M}_m^i(\tau)| \le \frac{C}{1-4\tau^2} + 4\sum_{i=1}^{m} \frac{1}{i^{2-\mu}}, \quad 0 \le \mu < 2; \tag{2.7}$$

$$\text{(ii)} \left|\sum_{i=1}^{2m+1} (m+1-i+\tau)\mathcal{M}_m^i(\tau)\right| \le \frac{C}{1-4\tau^2}. \tag{2.8}$$

**Proof.** Since the proof of the case where $m = 1$ is straightforward, we only consider the case where $m > 1$. Noting $|\tau| < 1/2$, we have

$$\begin{aligned} \sum_{i=1}^{2m+1} \frac{|m+1-i+\tau|^\mu}{|4(m+1-i+\tau)^2 - 1|} &= \sum_{i=0}^{m} \frac{|i+\tau|^\mu}{|4(i+\tau)^2 - 1|} + \sum_{i=1}^{m} \frac{|i-\tau|^\mu}{|4(i-\tau)^2 - 1|} \\ &\le \frac{|\tau|^\mu}{1-4\tau^2} + \frac{(1+\tau)^\mu}{4(1+\tau)^2 - 1} + \frac{(1-\tau)^\mu}{4(1-\tau)^2 - 1} \end{aligned}$$

$$+\sum_{i=2}^{m}\left[\frac{1}{2(i+\tau)^{2-\mu}}+\frac{1}{2(i-\tau)^{2-\mu}}\right]$$
$$\le \frac{13}{4(1-4\tau^2)}+\sum_{i=1}^{m-1}\frac{1}{i^{2-\mu}},\quad 0\le\mu<2$$

and

$$\sum_{i=1}^{2m+1}\frac{m+1-i+\tau}{4(m+1-i+\tau)^2-1}=\frac{\tau}{4\tau^2-1}+\sum_{i=1}^{m}\left[\frac{i+\tau}{4(i+\tau)^2-1}-\frac{(i-\tau)}{4(i-\tau)^2-1}\right]$$
$$=\frac{\tau}{4\tau^2-1}-\tau\sum_{i=1}^{m}\left[\frac{1}{(2i+1)^2-4\tau^2}+\frac{1}{(2i-1)^2-4\tau^2}\right].$$

By (2.4) and Lemma 2.2, we obtain (2.7) and (2.8), which completes the proof. □

**Lemma 2.4.** *For $\mathcal{M}_m^i(\tau)$ defined by (2.4),*

$$\left|\sum_{i=1}^{2m+1}\mathcal{M}_m^i\left(\pm\frac{1}{3}\right)\right|<\frac{C}{m}. \tag{2.9}$$

**Proof.** By (2.4) and straightforward calculation,

$$\sum_{i=1}^{2m+1}\mathcal{M}_m^i(\tau)=3\sum_{i=1}^{2m+1}\mathcal{H}_m^i(\tau)+\sum_{i=1}^{2m+1}\frac{1}{4(m+1-i+\tau)^2-1}$$
$$=3\sum_{i=1}^{2m+1}\mathcal{H}_m^i(\tau)-\frac{1}{2}\left(\frac{1}{2m+1+2\tau}+\frac{1}{2m+1-2\tau}\right),$$

which leads to (2.9) by noting the following inequality (cf. [30])

$$\left|\sum_{i=1}^{2m+1}\mathcal{H}_m^i\left(\pm\frac{1}{3}\right)\right|<\frac{C}{m}.$$

The proof is then complete. □

Let $f(t)\in C^{k+\alpha}[a,b](0<\alpha\le 1)$, $k$ be an integer and $\mathcal{S}_k(f):=\{\Pi_k f(t)\in C[a,b]\}$ be an interpolation function space in which $\Pi_k f(t)$ satisfies the following conditions:

(i) $\Pi_k f(t_i)=f(t_i),\ 0\le i\le n$;
(ii) $\Pi_k f(t)$ is a polynomial of degree k in $(t_{i-1},t_i)(1\le i\le n)$ and

$$\left|\frac{\mathrm{d}^l}{\mathrm{d}t^l}\left[f(t)-\Pi_k f(t)\right]\right|\le Ch^{k+\alpha-l},\quad l=0,1,\ldots,k.$$

**Lemma 2.5** (*[30], Lemma 3.3*). *Assume that $f(t)\in C^{k+\alpha}[a,b](0<\alpha\le 1)$ and $s\ne t_i$ for any $i=1,2,\ldots,n-1$. Let $\Pi_k f(t)\in\mathcal{S}_k(f)$ with $k\ge p$. Then*

$$\left|I_p(a,b;s,f)-I_p(a,b;s,\Pi_k f(t))\right|\le C\gamma^{-1}(h,s)h^{k+\alpha-p},\quad p=1,2, \tag{2.10}$$

*where*

$$\gamma(h,s)=\min_{0\le i\le n}\frac{|s-t_i|}{h}. \tag{2.11}$$

**Lemma 2.6.** *Assume $n = 2m + 1$. Then, under the same assumptions of Theorem 2.1,*

$$|I_2(a, b; s, f) - I_2(a, b; s, f_Q)| \le \begin{cases} Ch^{1+\alpha}, & f(t) \in C^{3+\alpha}[a, b], \\ C|\ln h|h^2, & f(t) \in C^4[a, b], \\ Ch^2, & f(t) \in C^{4+\alpha}[a, b], \end{cases} \tag{2.12}$$

*where* $0 < \alpha < 1$.

**Proof.** Let $f_C(t)$ be the piecewise cubic interpolant of $f(t)$, defined by

$$\begin{aligned} f_C(t) &= \frac{8(t - t_i)(t - t_{i-\frac{1}{2}})}{3h^3}[8(t - t_{i-1})f(t_{i-\frac{3}{4}}) - 3(t - t_{i-\frac{3}{4}})f(t_{i-1})] \\ &\quad + \frac{8(t - t_{i-1})(t - t_{i-\frac{3}{4}})}{3h^3}[(t - t_{i-\frac{1}{2}})f(t_i) - 6(t - t_i)f(t_{i-\frac{1}{2}})], \end{aligned} \tag{2.13}$$

where $t_{i-3/4} = t_{i-1} + h/4$, $t_{i-1/2} = t_{i-1} + h/2$, $t \in [t_{i-1}, t_i]$, $1 \le i \le n$. It is easy to see that $f_C(t) \in \mathcal{S}_3(f)$. Note that

$$\begin{aligned} f_Q(t) &= \frac{2(t - t_i)(t - t_{i-\frac{1}{2}})}{h^2} f(t_{i-1}) + \frac{2(t - t_{i-1})(t - t_{i-\frac{1}{2}})}{h^2} f(t_i) \\ &\quad + \frac{4(t - t_{i-1})(t - t_i)}{h^2} f(t_{i-\frac{1}{2}}), \quad t \in [t_{i-1}, t_i],\ 1 \le i \le n. \end{aligned} \tag{2.14}$$

Comparing (2.13) with (2.14), we find the relation between $f_C(t)$ and $f_Q(t)$:

$$f_C(t) - f_Q(t) = \beta_i (t - t_{i-1})(t - t_{i-\frac{1}{2}})(t - t_i) \tag{2.15}$$

with

$$\beta_i = \frac{8}{3h^3}[8f(t_{i-\frac{3}{4}}) - 3f(t_{i-1}) + f(t_i) - 6f(t_{i-\frac{1}{2}})]. \tag{2.16}$$

By straightforward calculation,

$$\begin{aligned} ⨍_a^b \frac{f_C(t) - f_Q(t)}{(t - s)^3} \mathrm{d}t &= \sum_{i=1, i \ne m+1}^{2m+1} \beta_i \int_{t_{i-1}}^{t_i} \frac{(t - t_{i-1})(t - t_{i-\frac{1}{2}})(t - t_i)}{(t - s)^3} \mathrm{d}t \\ &\quad + \beta_{m+1} ⨍_{t_m}^{t_{m+1}} \frac{(t - t_m)(t - t_{m+\frac{1}{2}})(t - t_{m+1})}{(t - s)^3} \mathrm{d}t \\ &= \sum_{i=1}^{2m+1} \beta_i \left[ 2h + 3(s - t_{i-\frac{1}{2}}) \ln \left| \frac{s - t_i}{s - t_{i-1}} \right| + \frac{(s - t_{i-\frac{1}{2}})^2 h}{(s - t_{i-1})(s - t_i)} \right] \\ &= h \sum_{i=1}^{2m+1} \beta_i \mathcal{M}_m^i \left( \pm \frac{1}{3} \right), \end{aligned} \tag{2.17}$$

where $s = t_m + h/2 \pm h/3$ has been used. If $f(t) \in C^{3+\alpha}[a, b] (0 < \alpha \le 1)$, by the Taylor expansion, we have

$$\beta_i = \frac{-f'''(\xi_i) + 3f'''(\theta_i) + f'''(\zeta_i)}{18} = O(h^\alpha) + \frac{f'''(s)}{6} + \frac{f'''(t_{i-\frac{1}{2}}) - f'''(s)}{6}, \tag{2.18}$$

where $\xi_i, \theta_i, \zeta_i \in \left[t_{i-1}, t_i\right]$. Substituting (2.18) into (2.17) gives

$$\begin{aligned} \left| ⨍_a^b \frac{f_C(t) - f_Q(t)}{(t - s)^3} \mathrm{d}t \right| &\le Ch^{1+\alpha} \sum_{i=1}^{2m+1} \left| \mathcal{M}_m^i \left( \pm \frac{1}{3} \right) \right| + \frac{h}{6} \left| f'''(s) \right| \left| \sum_{i=1}^{2m+1} \mathcal{M}_m^i \left( \pm \frac{1}{3} \right) \right| \\ &\quad + \frac{h^{1+\alpha}}{6} \sum_{i=1}^{2m+1} \left| m + 1 - i \pm \frac{1}{3} \right|^\alpha \left| \mathcal{M}_m^i \left( \pm \frac{1}{3} \right) \right|. \end{aligned}$$

By (2.7), (2.9) and the inequality

$$\frac{1}{2(m+1)} < 1 + \frac{1}{2} + \cdots + \frac{1}{m} - \ln m - \gamma < \frac{1}{2m},$$

where $\gamma$ is the Euler's constant, we get

$$\left| \fint_a^b \frac{f_C(t) - f_Q(t)}{(t-s)^3} \mathrm{d}t \right| \le Ch^{1+\alpha} \left( 1 + \sum_{i=1}^{m} \frac{1}{i^{2-\alpha}} \right) \le \begin{cases} Ch^{1+\alpha}, & 0 < \alpha < 1, \\ C|\ln h| h^2, & \alpha = 1. \end{cases} \tag{2.19}$$

If $f(t) \in C^{4+\alpha}[a, b](0 < \alpha < 1)$, (2.18) can be rewritten as

$$\beta_i = O(h) + \frac{f'''(s)}{6} + \frac{f^{(4)}(s)}{6}(t_{i-\frac{1}{2}} - s) + \frac{f^{(4)}(\vartheta_i) - f^{(4)}(s)}{6}(t_{i-\frac{1}{2}} - s), \tag{2.20}$$

where $\vartheta_i \in \left[s, t_{i-\frac{1}{2}}\right]$ or $\left[t_{i-\frac{1}{2}}, s\right]$. Thus, by Lemmas 2.3 and 2.4,

$$\begin{aligned} \left| \fint_a^b \frac{f_C(t) - f_Q(t)}{(t-s)^3} \mathrm{d}t \right| &\le Ch^2 \sum_{i=1}^{2m+1} \left| \mathcal{M}_m^i \left( \pm \frac{1}{3} \right) \right| + \frac{h}{6} \left| f'''(s) \right| \left| \sum_{i=1}^{2m+1} \mathcal{M}_m^i \left( \pm \frac{1}{3} \right) \right| \\ &\quad + \frac{h^2}{6} \left| f^{(4)}(s) \right| \left| \sum_{i=1}^{2m+1} \left( m + 1 - i \pm \frac{1}{3} \right) \mathcal{M}_m^i \left( \pm \frac{1}{3} \right) \right| \\ &\quad + Ch^{2+\alpha} \sum_{i=1}^{2m+1} \left| m + 1 - i \pm \frac{1}{3} \right|^{1+\alpha} \left| \mathcal{M}_m^i \left( \pm \frac{1}{3} \right) \right| \\ &\le Ch^2 + Ch^{2+\alpha} \sum_{i=1}^{m} \frac{1}{i^{1-\alpha}} \le Ch^2. \end{aligned} \tag{2.21}$$

Thus, (2.12) follows from the triangle inequality

$$\left| I_2(a, b; s, f) - I_2(a, b; s, f_Q) \right| \le |I_2(a, b; s, f) - I_2(a, b; s, f_C)| + \left| I_2(a, b; s, f_C) - I_2(a, b; s, f_Q) \right|,$$

Lemma 2.5, (2.19) and (2.21). The proof is complete. □

### 2.3. *The proof of* Theorem 2.1

Now we begin to prove our main result. If $m = 0$ or $m = n - 1$, (2.1) can be obtained directly from Lemma 2.5 by noting $\eta(s) = O(h^{-2})$. Thus we can assume that $1 \le m < n/2$ since the argument for the case $n/2 \le m < n - 1$ is similar. By definition (1.2), we have

$$I_2(a, b; s, f) - I_2(a, b; s, f_Q) = \fint_a^{t_{2m+1}} \frac{f(t) - f_Q(t)}{(t-s)^3} \mathrm{d}t + \int_{t_{2m+1}}^{b} \frac{f(t) - f_Q(t)}{(t-s)^3} \mathrm{d}t. \tag{2.22}$$

The first term can be estimated by Lemma 2.6, i.e.,

$$\left| \fint_a^{t_{2m+1}} \frac{f(t) - f_Q(t)}{(t-s)^3} \mathrm{d}t \right| \le \begin{cases} Ch^{1+\alpha}, & f(t) \in C^{3+\alpha}[a, b], \\ C|\ln h| h^2, & f(t) \in C^4[a, b], \\ Ch^2, & f(t) \in C^{4+\alpha}[a, b]. \end{cases} \tag{2.23}$$

As for the second term, by the standard interpolation theory,

$$\left| f(t) - f_Q(t) \right| \le Ch^3 \tag{2.24}$$

and consequently,

$$\left| \int_{t_{2m+1}}^{b} \frac{f(t) - f_Q(t)}{(t-s)^3} \mathrm{d}t \right| \le Ch^3 \int_{t_{2m+1}}^{b} \frac{1}{(t-s)^3} \mathrm{d}t \le C\eta(s) h^3. \tag{2.25}$$

Then, (2.1) follows immediately from (2.22), (2.23) and (2.25). □

### 2.4. The uniqueness of the superconvergence points

A natural question raised is whether the points $s = t_m + h/2 \pm h/3$ are those where the rate of convergence is maximized. To answer this question, we have to introduce some more notations and results.

Let $Q_n(x)$ be the function of the second kind associated with the Legendre polynomial $P_n(x)$, defined by (cf. [4])

$$Q_0(x) = \frac{1}{2}\ln\left|\frac{x+1}{x-1}\right|, \qquad Q_1(x) = xQ_0(x) - 1$$

and the recurrence relation

$$Q_{n+1}(x) = \frac{2n+1}{n+1}xQ_n(x) - \frac{n}{n+1}Q_{n-1}(x).$$

By straightforward calculation,

$$\mathcal{H}_m^i(\tau) = -Q_1(2m+2-2i+2\tau), \tag{2.26}$$

where $\mathcal{H}_m^i(\tau)$ is defined in (2.3). Define further that

$$\mathcal{W}(f;\tau) = f(2\tau) + \sum_{i=1}^{\infty}[f(2i+2\tau) + f(-2i+2\tau)], \quad |\tau| < \frac{1}{2}. \tag{2.27}$$

It is obvious that $\mathcal{W}$ is a linear operator on $f(x)$. In addition, we have the results below.

**Lemma 2.7.** *For the operator $\mathcal{W}$ defined by (2.27), there holds the identity*

$$\mathcal{W}(Q_1;\tau) = -\ln[2\cos(\tau\pi)], \quad |\tau| < \frac{1}{2}. \tag{2.28}$$

**Proof.** For $|\tau| < 1/2$, we have

$$\begin{aligned}\mathcal{W}(Q_0;\tau) &= \frac{1}{2}\ln\frac{1+2\tau}{1-2\tau} + \frac{1}{2}\sum_{i=1}^{\infty}\left(\ln\frac{2i+1+2\tau}{2i-1+2\tau} + \ln\frac{2i-1-2\tau}{2i+1-2\tau}\right)\\ &= \lim_{n\to\infty}\frac{1}{2}\ln\frac{2n+1+2\tau}{2n+1-2\tau} = 0\end{aligned}$$

and by the well-known identity (cf. [3], (1.2.7))

$$\lim_{n\to\infty}\sum_{i=-n}^{n}\frac{1}{i+\frac{1}{2}-x} = \pi\tan(\pi x), \tag{2.29}$$

we have further

$$\begin{aligned}\mathcal{W}(xQ_0';\tau) &= \frac{2\tau}{1-4\tau^2} + \sum_{i=1}^{\infty}\left[\frac{2i+2\tau}{1-(2i+2\tau)^2} + \frac{-2i+2\tau}{1-(-2i+2\tau)^2}\right]\\ &= \sum_{i=1}^{\infty}\left(\frac{1}{2i-1-2\tau} + \frac{1}{-2i+1-2\tau}\right)\\ &= \frac{1}{2}\lim_{n\to\infty}\sum_{i=-n}^{n}\frac{1}{i+\frac{1}{2}-\tau}\\ &= \frac{\pi}{2}\tan(\tau\pi).\end{aligned}$$

Then

$$\mathcal{W}(Q_1';\tau) = \mathcal{W}(Q_0 + xQ_0';\tau) = \mathcal{W}(Q_0;\tau) + \mathcal{W}(xQ_0';\tau) = \frac{\pi}{2}\tan(\tau\pi), \tag{2.30}$$

which implies

$$\mathcal{W}(Q_1;\tau)=\int \pi\tan(\tau\pi)\mathrm{d}\tau=-\ln[\cos(\tau\pi)]+C. \tag{2.31}$$

What remains is to determine the constant $C$. By using the identities (cf. [3], Chapter 1, Section 1.2, and [25], Chapter II, Section 4),

$$\begin{aligned} x\cot x &= 1+\sum_{k=1}^{\infty}(-1)^k B_{2k}\frac{(2x)^{2k}}{(2k)!},\\ \ln(2\sin x) &= -\sum_{j=1}^{\infty}\frac{1}{j}\cos(2jx), \quad x\in(0,\pi), \end{aligned} \tag{2.32}$$

where $\{B_{2k}\}$ denote the Bernoulli numbers, we have

$$\sum_{k=1}^{\infty}(-1)^k B_{2k}\frac{(2x)^{2k+1}}{(2k+1)!}=2x\ln(\sin x)+2\left[(\ln 2-1)x+\sum_{j=1}^{\infty}\frac{1}{2j^2}\sin(2jx)\right].$$

Setting $x=\pi/2$ gives

$$\sum_{k=1}^{\infty}\frac{(-1)^k B_{2k}\pi^{2k}}{(2k+1)!}=\ln 2-1.$$

Thus,

$$\begin{aligned} \mathcal{W}(Q_1;0) &= -1+2\sum_{i=1}^{\infty}Q_1(2i)=-1+2\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{(2k+1)(2i)^{2k}}\\ &= -1+\sum_{k=1}^{\infty}\frac{(-1)^{k+1}B_{2k}\pi^{2k}}{(2k+1)!}=-\ln 2, \end{aligned} \tag{2.33}$$

where we have used the formulae (cf. [4] and [3], Chapter 1, Section 1.2)

$$\begin{aligned} Q_1(x) &= \sum_{k=1}^{\infty}\frac{1}{(2k+1)x^{2k}}, \quad |x|>1,\\ \sum_{i=1}^{\infty}\frac{1}{i^{2k}} &= \frac{(-1)^{k+1}2^{2k-1}}{(2k)!}B_{2k}\pi^{2k}. \end{aligned} \tag{2.34}$$

By (2.31) and (2.33) we obtain (2.28), which completes the proof. □

Now, we investigate the convergence rate of Simpson's rule (1.6) and for simplicity of presentation, confine ourselves to the special case where $f(t)=t^3$. Suppose that $n=2m+1$ and $s=t_m+h/2+\tau h(|\tau|<1/2)$. Since $f(t)$ is identical to $f_C(t)$ defined by (2.13) in this case, by (2.17) and through direct calculation, we have

$$\begin{aligned} \oint_a^b \frac{f(t)-f_Q(t)}{(t-s)^3}\mathrm{d}t &= \oint_a^b \frac{f_C(t)-f_Q(t)}{(t-s)^3}\mathrm{d}t = h\sum_{i=1}^{2m+1}\mathcal{M}_m^i(\tau)\\ &= 3h\sum_{i=1}^{2m+1}\mathcal{H}_m^i(\tau)-\frac{h}{2}\mathcal{A}_m(\tau), \end{aligned}$$

where

$$\mathcal{A}_m(\tau)=\frac{1}{2m+1+2\tau}+\frac{1}{2m+1-2\tau}.$$

By (2.26), (2.27) and Lemma 2.7, we get

$$\begin{aligned}\oint_a^b \frac{f(t)-f_Q(t)}{(t-s)^3}\mathrm{d}t &= -3h\sum_{i=1}^{2m+1} Q_1(2m+2-2i+2\tau)-\frac{h}{2}\mathcal{A}_m(\tau)\\ &= -3h\left\{Q_1(2\tau)+\sum_{i=1}^{m}[Q_1(2i+2\tau)+Q_1(-2i+2\tau)]\right\}-\frac{h}{2}\mathcal{A}_m(\tau)\\ &= 3h\ln[2\cos(\tau\pi)]+3h\sum_{i=m+1}^{\infty}[Q_1(2i+2\tau)+Q_1(-2i+2\tau)]-\frac{h}{2}\mathcal{A}_m(\tau).\end{aligned} \tag{2.35}$$

By the classical identity [4]

$$Q_n(x)=\frac{1}{2^{n+1}}\int_{-1}^{1}\frac{(1-t^2)^n}{(x-t)^{n+1}}\mathrm{d}t, \quad |x|>1, n=0,1,2,\ldots, \tag{2.36}$$

we get

$$|Q_n(x)|\le\frac{C}{(|x|-1)^{n+1}}, \quad |x|>1,$$

which leads to

$$\left|\sum_{i=m+1}^{\infty}[Q_1(2i+2\tau)+Q_1(-2i+2\tau)]\right|\le C\sum_{i=m+1}^{\infty}\left[\frac{1}{(2i-1+2\tau)^2}+\frac{1}{(2i-1-2\tau)^2}\right]\le Ch. \tag{2.37}$$

Substituting this estimate into (2.35) and by noting the definition of $\mathcal{A}_m(\tau)$, we finally obtain

$$\oint_a^b \frac{f(t)-f_Q(t)}{(t-s)^3}\mathrm{d}t = 3h\ln[2\cos(\tau\pi)]+O(h^2), \tag{2.38}$$

for $f(t)=t^3$.

For a general function $f(t)\in C^{3+\alpha}[a,b](0<\alpha\le 1)$, by a similar argument, we can get

$$\oint_a^b \frac{f(t)-f_Q(t)}{(t-s)^3}\,\mathrm{d}t = \frac{h}{2}f'''(s)\ln[2\cos(\tau\pi)]+O(h^{1+\alpha}). \tag{2.39}$$

One can see that the first-order term in the error vanishes if and only if $\tau=\pm 1/3$. In other words, the convergence rate will be $O(h)$ if the singular point does not coincide with the superconvergence points given in Theorem 2.1. Thus, our superconvergence points maximize the convergence rate of Simpson's rule (1.6).

## 3. Some applications

In this section, we consider some applications of the superconvergence result described in the above section. We first apply this result to the evaluation of the finite-part integral with a third-order singularity in the general case where the singular point is not a superconvergence point. Secondly, we use the result to solve a certain integral equation possessing a third-order singularity.

### *3.1. Application to the evaluation of finite-part integral*

By Theorem 2.1, in using Simpson's rule (1.6) to evaluate (1.1) with $p=2$, one can expect a second-order accuracy when the singular point happens to be a superconvergence point away from the interval endpoints. Generally speaking, the singular point is probably not a superconvergence point and the accuracy can only be $O(h)$ (cf. [9]). Here, based on the superconvergence result, we suggest two algorithms through two different approaches to get a second-order accuracy for the general case.

In the first approach, we translate the original mesh such that the singular point is a superconvergence point with respect to the resulting mesh. More explicitly, we have

**Algorithm 1.** Let $a = t_0 < t_1 < \cdots < t_n = b$ be the original uniform mesh. Translate the interior mesh points to get a new mesh $a = t'_0 < t'_1 < \cdots < t'_n = b$ in which the singular point $s$ is a superconvergence point. Finally, use Simpson's rule (1.6) on this new mesh to approximate $I_2(a, b; s, f)$.

We see that the new mesh in Algorithm 1 is still uniform except two shorter or longer subintervals near the endpoints of the interval $[a, b]$. The superconvergence analysis performed in the above section can be easily extended to this kind of grid and therefore, a second-order accuracy can still be expected.

The second approach is not as straightforward as the first one. However, this approach does not require the change of the mesh and thus keeps the main advantage of the composite Newton–Cotes method. We recall that Wu and Yu [31] suggested an indirect method for (1.1) with $p = 1$ and Du [9] discussed a similar method for the case $p = 2$. The key point of their methods is based on the fact that one can always expect a good accuracy when the singular point is the midpoint of a subinterval. Here we suggest a new indirect method based on the superconvergence result. More interesting is that the accuracy of our indirect method is one order higher than that of Du's [9] with the same computational cost.

**Algorithm 2.** Assume that $s_1$ and $s_2$ are two superconvergence points nearest to $s$ such that $s_1 \leq s \leq s_2$. Then, approximate $I_2(a, b; s, f)$ by

$$I_2^*(a, b; s, f_Q) = \frac{1}{s_2 - s_1}\left[(s - s_1)I_2(a, b; s_2, f_Q) + (s_2 - s)I_2(a, b; s_1, f_Q)\right], \tag{3.1}$$

where $I_2(a, b; s_i, f_Q)(i = 1, 2)$ are given by (1.6).

By using $I_2^*(a, b; s, f_Q)$, instead of $I_2(a, b; s, f_Q)$, to approximate $I_2(a, b; s, f)$, we can always expect a second-order accuracy under certain conditions, which is stated in Theorem 3.3. We start our analysis by first giving two Lemmas.

**Lemma 3.1** (*[9], Lemma 2.4*). *If $f(t) \in C^k[a, b]$ $(k \geq 3)$, then $I_2(a, b; s, f)$, as a function of $s$, belongs to $C^{k-3}(a, b)$.*

**Lemma 3.2** (*[9], Lemma 2.5*). *If $f(t) \in C^k[a, b]$ $(k \geq 3)$, then there exists a positive constant $C = C(a, b, l, f)$, depending only on $a, b, l$ and $f$, such that*

$$\left|\frac{\mathrm{d}^l}{\mathrm{d}s^l} I_2(a, b; s, f)\right| \leq C\varrho^{-l-2}(s), \tag{3.2}$$

*where $\varrho(s) = \min\{s - a, b - s\}$, $l = 1, 2, \ldots, k - 3$.*

**Theorem 3.3.** *Suppose that $f(t) \in C^5[a, b]$ and the mesh is uniform. Let $I_2^*(a, b; s, f_Q)$ be computed by (3.1) with $s_1$ and $s_2$ being two superconvergence points nearest to $s$ and $s_1 \leq s \leq s_2$. Then*

$$\left|I_2(a, b; s, f) - I_2^*(a, b; s, f_Q)\right| \leq C\varrho^{-4}(s)h^2. \tag{3.3}$$

**Proof.** If $s_1 = s$ or $s_2 = s$, then (3.3) is obvious by Theorem 2.1 and by noting $\eta(s) = \varrho^{-2}(s)$. Now consider the more general case where $s_1 < s < s_2$. Set

$$I_2^*(a, b; s, f) = \frac{1}{s_2 - s_1}\{(s - s_1)I_2(a, b; s_2, f) + (s_2 - s)I_2(a, b; s_1, f)\}.$$

On the one hand, following Theorem 2.1, we have

$$\begin{aligned}\left|I_2^*(a, b; s, f_Q) - I_2^*(a, b; s, f)\right| &\leq \left|I_2(a, b; s_2, f_Q) - I_2(a, b; s_2, f)\right| \\ &\quad + \left|I_2(a, b; s_1, f_Q) - I_2(a, b; s_1, f)\right| \\ &\leq C[1 + \eta(s_2)h]h^2 + C[1 + \eta(s_1)h]h^2 \\ &\leq C[1 + \eta(s)h]h^2.\end{aligned} \tag{3.4}$$

On the other hand, since $f(t) \in C^5[a,b]$, $I(a,b;s,f)$, being the function of $s$, belongs to $C^2(a,b)$ due to Lemma 3.1. Note that $I_2^*(a,b;s,f)$ is actually the linear interpolant of $I_2(a,b;s,f)$ with respect to $s$. Thus,

$$\left|I_2(a,b;s,f) - I_2^*(a,b;s,f)\right| \le \max_{s\in[s_1,s_2]} \left|\frac{\mathrm{d}^2}{\mathrm{d}s^2} I_2(a,b;s,f)\right| h^2 \le C\varrho^{-4}(s)h^2. \tag{3.5}$$

Finally, (3.3) is obtained by (3.4) and (3.5) and the triangular inequality. □

It is obvious that Algorithm 2 doubles the work by comparison with Algorithm 1. The difference between Algorithm 2 and Du's indirect method (cf. [9]) consists in the way to choose $s_1$ and $s_2$. In Du's method, $s_1$ and $s_2$ are chosen to be two subinterval midpoints nearest to $s$ and the accuracy is $O(h)$. Therefore, the computational cost for both indirect methods is the same while Algorithm 2 is of second order.

### *3.2. Application to the solution of finite-part integral equation*

Another important application is to solve the finite-part integral equation using the superconvergence result. Finite-part integral equations often arise in some numerical analysis of partial different equation and many physical problems, and there exist a lot of numerical methods to solve them. In [32], the so-called natural boundary integral method reduces the boundary value problem of partial differential equation into a finite-part integral equation on the boundary by using Green's function and Green's formula, and then Galerkin methods are applied to solve it. In [19], a fully discrete method is described for the numerical solution of the finite-part integral equation arising from the combined double- and single-layer approach for the solution of the exterior Neumann problem for the two-dimensional Helmholtz equation in smooth domains. Moreover, the numerical solution of a non-linear finite-part boundary integral equation is discussed in [12], and a direct finite-part integral approach is considered to solve harmonic problems with non-linear boundary conditions by using a practical variant of the Galerkin boundary element method.

For simplicity, here we consider the following Fredholm integral equation of the first kind with a third-order singularity kernel:

$$\int\!\!\!\!\!\!=_{-1}^{1} \frac{u(t)}{(t-s)^3}\mathrm{d}t = g(s), \tag{3.6}$$

with the boundary condition $u(-1) = u(1) = 0$, where $g(s)$ is a given function. By using the composite Simpson's rule (1.6) to approximate the finite-part integral in (3.6) and by using the collocation procedure, we get the linear system

$$\sum_{j=1}^{2n-1} \omega_j(s_i)u_h(t_{\frac{j}{2}}) = g(s_i), \tag{3.7}$$

where $s_i$ $(1 \le i \le 2n-1)$ denote the collocation points, $u_h$ is the collocation solution and the homogeneous boundary condition has been used. To the best of our knowledge, there has been no criterion in choosing those $2n-1$ collocation points. If the collocation points are chosen arbitrarily, the truncation error for (3.7) will be $O(h)$ in general. Due to the superconvergence analysis for the composite Simpson's rule, it is now natural for us to choose the superconvergence points to be the collocation points. We see from Theorem 2.1 that there are two superconvergence points in each subinterval and $2n$ such superconvergence points in the whole interval. Then, by choosing any $2n-1$ superconvergence points to be the collocation points, we get a special collocation system. This choice of the collocation points is surely expected to produce a better result than other choices. Actually the numerical results given in the next section show that a certain discrete $L^2$-error and the global maximal nodal error are both one and a half order higher than that of other choices of the collocation points. The superconvergence analysis for the finite-part integral equation is beyond the aim and scope of the present paper and will be given elsewhere.

## 4. Numerical experiments

Here we present several examples to confirm the superconvergence result and to test our algorithms in Section 3. In the first three examples, Simpson's rule (1.6) is used to evaluate certain finite-part integrals in the form of (1.1) with

Table 1
Errors for the case where $s = t_{[n/4]} + h/2 + \tau h$

| $n$ | $\tau = -1/3$ | $\tau = 1/3$ | $\tau = 0$ | $\tau = 1/4$ |
|---|---|---|---|---|
| 128 | 5.23091E−4 | 5.23125E−4 | 6.39672E−2 | 3.15816E−2 |
| 256 | 1.30776E−4 | 1.30778E−4 | 3.22374E−2 | 1.60182E−2 |
| 512 | 3.26942E−5 | 3.26944E−5 | 1.61822E−2 | 8.06596E−3 |
| 1024 | 8.17359E−6 | 8.17360E−6 | 8.10695E−3 | 4.04720E−3 |
| 2048 | 2.04331E−6 | 2.04339E−6 | 4.05744E−3 | 2.02715E−3 |
| 4096 | 5.10673E−7 | 5.10609E−7 | 2.02971E−3 | 1.01446E−3 |
| Error order | $h^{2.000}$ | $h^{2.000}$ | $h^{0.996}$ | $h^{0.992}$ |

Table 2
Errors for the case where $s = t_0 + h/2 + \tau h$

| $n$ | $\tau = -1/3$ | $\tau = 1/3$ | $\tau = 0$ | $\tau = 1/4$ |
|---|---|---|---|---|
| 128 | 2.45370E−2 | 1.22033E−3 | 1.26449E−1 | 6.31565E−2 |
| 256 | 1.23589E−2 | 4.77946E−4 | 6.34829E−2 | 3.18075E−2 |
| 512 | 6.20211E−3 | 2.05917E−4 | 3.18060E−2 | 1.59611E−2 |
| 1024 | 3.10686E−3 | 9.46916E−5 | 1.59192E−2 | 7.99486E−3 |
| 2048 | 1.55190E−3 | 4.53250E−5 | 7.96361E−3 | 4.00101E−3 |
| 4096 | 8.23796E−4 | 2.22144E−5 | 3.98274E−3 | 2.00141E−3 |
| Error order | $h^{0.979}$ | $h^{1.156}$ | $h^{0.998}$ | $h^{0.996}$ |

$p = 2$. The last example is to solve a finite-part integral equation (3.6). Throughout this section, all the computation is performed in double precision and all the numerical estimates of the error orders presented in the last rows of the tables are calculated by a least squares fit.

**Example 4.1.** Consider the finite-part integral (1.1) with a smooth integral function $f(t) = t^4$ and $a = -b = -1$. The exact value of this finite-part integral is

$$I_2(-1, 1; s, t^4) = 6s - \frac{8s^3 - 6s^5}{(1 - s^2)^2} + 6s^2 \ln\left|\frac{1 - s}{1 + s}\right|.$$

We adopt a uniform mesh and examine the accuracy for two types of dynamic singular points $s = t_{[n/4]} + h/2 + \tau h$ and $s = t_0 + h/2 + \tau h$. The errors are presented in Table 1 and Table 2, respectively. Table 1 shows that the accuracy for the superconvergence points $\tau = \pm 1/3$ is $O(h^2)$, which agrees quite well with the superconvergence estimate in Theorem 2.1. One can see from the same table that the accuracy for the two non-superconvergence points, including the subinterval midpoint $\tau = 0$, is only $O(h)$. As for Table 2, the accuracy for the four points are all $O(h)$, which implies that the superconvergence phenomenon disappears. However, by noting that the point $s$ is very close to an endpoint and $\eta(s) = O(h^{-2})$ in this case, we claim that the results in Table 2 still agree with the estimate (2.1). Furthermore, the accuracy at the superconvergence points is still much better than that at the non-superconvergence points, although the convergence order is the same.

**Example 4.2.** Now we consider an example with less regularity. Let $a = -b = -1$, $s = 0$ and

$$f(t) = \mathcal{F}_i(t) := t^3 + (2 + \mathrm{sign}(t))|t|^{3-i+1/2}, \quad i = 0, 1.$$

Obviously, $\mathcal{F}_i(t) \in C^{3-i+1/2}[-1, 1]$ $(i = 0, 1)$. The exact value of the finite-part integral is

$$I_2(-1, 1; 0, \mathcal{F}_i(t)) = \frac{10 - 4i}{3 - 2i}.$$

Here we employ two meshes, Mesh I and Mesh II. In Mesh I, the singular point $s = 0$ is always located at the superconvergence point $\tau = -1/3$. In Mesh II, $s$ is located at the midpoint of a certain subinterval. Both meshes are uniform except two shorter or longer subintervals near the endpoints $-1$ and 1. The corresponding numerical results are given in Table 3. For $\mathcal{F}_0(t)$, we get the desired superconvergence error bound $O(h^{3/2})$ on Mesh I and a general

Table 3
The errors for the cases where $f(t) = \mathcal{F}_i(t)$

| $n$ | $\mathcal{F}_0(t)$ | | $\mathcal{F}_1(t)$ | |
|---|---|---|---|---|
| | Mesh I | Mesh II | Mesh I | Mesh II |
| 128 | 7.78225E−3 | 3.51222E−2 | 2.69268E−2 | 5.66083E−1 |
| 256 | 2.75144E−3 | 1.71758E−2 | 1.90401E−2 | 3.93552E−1 |
| 512 | 9.72781E−4 | 8.45168E−3 | 1.34634E−2 | 2.74919E−1 |
| 1024 | 3.43930E−4 | 4.17768E−3 | 9.52007E−3 | 1.92715E−1 |
| 2048 | 1.21598E−4 | 2.07181E−3 | 6.73170E−3 | 1.35429E−1 |
| 4096 | 4.29910E−5 | 1.02989E−3 | 4.76003E−3 | 9.53420E−2 |
| Error order | $h^{1.500}$ | $h^{1.018}$ | $h^{0.500}$ | $h^{0.514}$ |

Table 4
Numerical results for $I_2(-1, 1; 0.5, \cos t)$

| $n$ | Algorithm 2 | | Du's method | |
|---|---|---|---|---|
| | $I_2^\star$ | Error | $I_2^\star$ | Error |
| 31 | 0.318668 | 0.001921 | 0.302890 | 0.017699 |
| 61 | 0.320041 | 0.000548 | 0.313083 | 0.007506 |
| 121 | 0.320447 | 0.000142 | 0.317333 | 0.003256 |
| 241 | 0.320553 | 0.000036 | 0.319083 | 0.001506 |
| 481 | 0.320580 | 0.000009 | 0.319867 | 0.000722 |
| 961 | 0.320587 | 0.000002 | 0.320236 | 0.000353 |

Table 5
$L^2$ and $L^\infty$-error for solving (3.6) by using the composite Simpson's rule

| $n$ | $\mathcal{S}_1$ | | $\mathcal{S}_2$ | |
|---|---|---|---|---|
| | $e_2$ | $e_\infty$ | $e_2$ | $e_\infty$ |
| 32 | 6.76442E−4 | 5.73493E−4 | 9.87349E−3 | 9.02045E−3 |
| 64 | 1.27452E−4 | 1.08542E−4 | 4.73014E−3 | 4.36806E−3 |
| 128 | 2.33787E−5 | 1.99946E−5 | 2.33866E−3 | 2.16710E−3 |
| 256 | 4.22783E−6 | 3.62812E−6 | 1.16849E−3 | 1.08364E−3 |
| Error order | $h^{2.467}$ | $h^{2.462}$ | $h^{1.001}$ | $h^{1.000}$ |

error bound $O(h)$ on Mesh II, which is in good agreement with our theoretical analysis. As for $\mathcal{F}_1(t)$, The accuracy on Mesh I and Mesh II are all $O(h^{1/2})$, which implies that the regularity assumption on $f(t)$ in the superconvergence estimate (2.1) cannot be weakened.

**Example 4.3.** In this example we investigate the validation of the two algorithms given in the above section for the evaluation of finite-part integrals. The results for $\mathcal{F}_0(t)$ on Mesh I in Table 3 confirms that Algorithm 1 produces the desired second-order accuracy when the singular point is not a superconvergence point. Here we only need to investigate the performance of Algorithm 2 and that of Du's (given by Theorem 3.3 in [9]). Let $f(t) = \cos t$ and compute $I_2(-1, 1; 0.5, \cos t)$, which is also evaluated in [9]. The exact value of this finite-part integral is 0.320589 (exact to $10^{-6}$). The approximate values for this integral are given in Table 4, which confirms the fact that Algorithm 2 is a second-order method while Du's method is a first-order one.

**Example 4.4.** Finally, we solve the finite-part integral equation (3.6) with the homogeneous boundary condition and

$$g(s) = 30s^3 - 14s + (15s^4 - 12s^2 + 1)\ln\frac{1-s}{1+s}.$$

The exact solution is $u(t) = t^2(t^2-1)^2$. We adopt a uniform mesh and get the linear system (3.7) with two sets of collocation points:

$$\mathcal{S}_1 = \{t_i + h/2 \pm h/3, 0 \le i < n-1\} \cup \{t_{n-1} + h/2 - h/3\},$$

Table 6
$L^2$ and $L^\infty$-error for solving (3.6) by using composite trapezoidal rule

| $n$ | $\mathcal{S}_3$ | | $\mathcal{S}_4$ | |
|---|---|---|---|---|
| | $e_2^*$ | $e_\infty^*$ | $e_2^*$ | $e_\infty^*$ |
| 32 | 1.76665E−2 | 1.44663E−2 | 0.10503 | 0.12079 |
| 64 | 6.10924E−3 | 5.08831E−3 | 0.10447 | 0.12162 |
| 128 | 2.13211E−3 | 1.79699E−3 | 0.10730 | 0.12487 |
| 256 | 7.48437E−4 | 6.35999E−4 | 0.10990 | 0.12756 |
| Error order | $h^{1.498}$ | $h^{1.498}$ | – | – |

$$\mathcal{S}_2 = \{t_i + h/2 \pm h/4, 0 \le i < n-1\} \cup \{t_{n-1} + h/2 - h/4\},$$
$$\mathcal{S}_3 = \{t_i + h/2,\ 0 \le i < n-1\},$$
$$\mathcal{S}_4 = \{t_i + h/4,\ 0 \le i < n-1\}.$$

Obviously, $\mathcal{S}_1$ consists of the superconvergence points for Simpson's rule while $\mathcal{S}_2$ does not, and $\mathcal{S}_3$ consists of the superconvergence points for the trapezoidal rule (cf. [30]) while $\mathcal{S}_4$ does not. We examine the following discrete $L^2$-error and the global maximal nodal error

$$e_2 = \left( \sum_{1 \le i \le 2n-1} |u(x_i) - u_h(x_i)|^2 h \right)^{1/2},$$
$$e_\infty = \max_{1 \le i \le 2n-1} |u(x_i) - u_h(x_i)|,$$
$$e_2^* = \left( \sum_{1 \le i \le n-1} |u(x_i) - u_h(x_i)|^2 h \right)^{1/2},$$
$$e_\infty^* = \max_{1 \le i \le n-1} |u(x_i) - u_h(x_i)|.$$

Here, $u_h(x_i)$ denotes the numerical solution at $x_i$. Numerical results are presented in Table 5 for Simpson's rule and in Table 6 for trapezoidal rule. We can see that the accuracy is much improved if we collocate the integral equation at the superconvergence points. And clearly, the rate of convergence for Simpson's rule is one order higher than that for trapezoidal rule.

## Acknowledgements

The work of X.P. Zhang and D.H. Yu was supported in part by the National Basic Research Program of China (No. 2005CB321701), the National Natural Science Foundation of China (No. 10531080) and the Natural Science Foundation of Beijing (No. 1072009). The work of J.M. Wu was supported by the National Natural Science Foundation of China (No. 10671025).

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