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Nodal-type collocation methods for hypersingular integral equations and nonlocal diffusion problems☆

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Abstract

In this paper, we study nodal-type quadrature rules for approximating hypersingular integrals and their applications to numerical solution of finite-part integral equations and nonlocal diffusion problems. We first derive explicit expressions for the quadrature coefficients and establish corresponding error estimates. Some collocation schemes are then constructed based on these rules to numerically solve certain type of finite-part integral equations and nonlocal diffusion problems in one dimension, and condition number and optimal error estimates for the proposed schemes are also rigorously obtained. On uniform grids, these schemes are of Toeplitz structure which results in many advantages in developing fast linear solvers. Various numerical experiments are also performed to illustrate the theoretical results.

Keywords: Collocation method; Hypersingular integrals; Nonlocal diffusion; Nodal-type quadrature rules

1. Introduction

Among various techniques for solving integral equations, collocation methods are among the simplest. It is well known that the performance of a collocation method depends crucially on the quadrature rule used for approximating the integrals. Classic quadrature rules and the corresponding rigorous analysis have been thoroughly researched for traditional Riemann integrals. However, these rules often cannot be directly applied to integrals of the

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type

$$\mathcal{K}_s u(x) = \fint_a^b \frac{u(y)}{|y-x|^{1+2s}}\,dy, \quad x \in (a,b),\ s \in [0,1), \tag{1}$$

due to the strong singularity of the kernel function involved. The integral in (1) is divergent in the classic Riemann sense, and should be understood in the Hadamard finite part sense. These types of integrals are often referred to as "Hadamard finite part integrals" or "hypersingular integrals". There are several equivalent mathematically definitions for these finite-part integrals in the literature [1]. Here, we adopt the following definition: for $x \in (a,b)$,

$$\mathcal{K}_s u(x) = \begin{cases} \lim\limits_{\epsilon\to 0}\left(\displaystyle\int_{\Omega_\epsilon} \frac{u(y)}{|y-x|}\,dy + 2u(x)\ln\epsilon\right), & s = 0, \\ \lim\limits_{\epsilon\to 0}\left(\displaystyle\int_{\Omega_\epsilon} \frac{u(y)}{|y-x|^{1+2s}}\,dy - \frac{u(x)}{s\epsilon^{2s}}\right), & s \in (0,1), \end{cases} \tag{2}$$

where $\Omega_\epsilon = (a,b)\backslash(x-\epsilon, x+\epsilon)$. A function $u(x)$ is said to be Hadamard finite-part integrable with respect to the weight $|y-x|^{-2s-1}$ if the limit on the right-hand side of (2) exists [1]. Assuming u is absolutely integrable on (a,b), then a sufficient condition for $u(x)$ to be finite-part integrable is that $u(x)$ is α-Hölder continuous for some $\alpha \in (2s, 1)$ on (a,b) if $s \in [0, 0.5)$, and $u'(x)$ is α-Hölder continuous for some $\alpha \in (2s-1, 1)$ on (a,b) if $s \in [0.5, 1)$.

A related topic is the study of quadrature rules for hypersingular integrals. Numerous researches have been devoted to this purpose, such as Gaussian-type methods [2–5], Newton–Cotes type methods [6,7], transformation methods [8,9], and other methods [10]. However, most of these works are limited to the situation $s = 0.5$. Hypersingular integrals appear frequently in many physical problems, such as electromagnetic scattering, fracture mechanics, continuum mechanics and so on [11–14]. The peridynamics (PD) model, initially proposed by Silling [13], is an integral-type nonlocal continuum model for the mechanics of materials, which provides an alternative setup to that of classic continuum mechanics based on PDEs. As shown in [15–17], linear scalar steady-state PD operators share similarities with nonlocal diffusion (ND) operators, which make the study of PD and general nonlocal diffusion models relevant to each other. In fact, both PD and ND problems can be regarded as Fredholm singular integral equations of the second kind. When collocation schemes are applied for numerically approximating solutions of these problems, it is natural to choose nodal points as the collocation points. In [6], Wu and Sun proposed many kinds of nodal-type schemes, with Toeplitz-type structures, for the integral operator (1) with $s = 0.5$; those schemes were applied successfully to electromagnetic scattering from cavities. Tian et al. discussed a fast collocation method for a nonlocal diffusion model in [18], but their work is restricted to the case $s = 0$. The work of this paper is inspired by [6] and [18]. We extend the nodal-type quadrature rules based on piecewise linear approximations for computing (1) from $s = 0.5$ to $s \in [0, 1)$, and the developed rules are valid for both uniform and nonuniform grids. Collocation schemes based on these rules for solving finite-part integral equations and nonlocal diffusion models are also considered.

The rest of the paper is organized as follows. In Section 2, we propose nodal-type quadrature rules for evaluating Hadamard finite-part integrals, including explicit expressions for the quadrature coefficients; a corresponding error analysis on general grids is also provided. Based on these quadrature rules, collocation schemes are then developed, in Sections 3 and 4, to approximate certain finite-part integral equations and nonlocal diffusion models in one-dimension respectively, where condition number estimates and the optimal error analysis are obtained for both problems on uniform grids. In Section 5, various numerical test results are reported to demonstrate the efficiency and accuracy of the proposed schemes. Concluding remarks are given in Section 6.

2. Nodal-type quadrature rules for finite-part integrals

Consider, in one dimension, the partition of the interval $[a, b]$

$$\pi_h : a = x_0 < x_1 < \cdots < x_n < x_{n+1} = b \tag{3}$$

with grid sizes $h_i = x_i - x_{i-1}$, $i = 1, 2, \ldots, n+1$. Let S^h be the space of continuous piecewise-linear polynomials defined with respect to the partition (3) and choose the standard "hat" functions as a basis which we denote as

$\{\phi_j(x)\}_{j=0}^{n+1}$. Substituting $\Pi_h u(x) = \sum_{j=0}^{n+1} u(x_j)\phi_j(x)$ in (1) yields the nodal-type quadrature rule for approximating the value of (1) at the nodal points x_i given by

$$\mathcal{K}_s u(x_i) \approx \mathcal{Q}_h^{(s)} u(x_i) = \sum_{j=0}^{n+1} \alpha_{ij}^{(s)} u(x_j), \tag{4}$$

where

$$\alpha_{ij}^{(s)} = \,⨎_{x_{j-1}}^{x_{j+1}} \frac{\phi_j(y)}{|y - x_i|^{1+2s}}\, dy. \tag{5}$$

The definition (2) is generally insufficient to obtain the explicit expression of $\alpha_{ij}^{(s)}$, and we need introduce a linear operator $\mathcal{K}_s^*$ by

$$\mathcal{K}_s^* u(x) = \lim_{\epsilon \to 0} \left(\int_{\Omega_\epsilon} \frac{u(y)}{|y - x|^{1+2s}}\, dy + r(\epsilon) \right) \tag{6}$$

where

$$r(\epsilon) = \begin{cases} \ln \epsilon \left[u(x^-) + u(x^+) \right], & s = 0, \\ \dfrac{\epsilon^{-2s}}{-2s} \left[u(x^-) + u(x^+) \right], & s \in (0, 0.5), \\ (-\epsilon^{-1}) \left[u(x^-) + u(x^+) \right] - \ln \epsilon \left[u'(x^-) - u'(x^+) \right], & s = 0.5, \\ \dfrac{\epsilon^{-2s}}{-2s} \left[u(x^-) + u(x^+) \right] - \dfrac{\epsilon^{1-2s}}{1 - 2s} \left[u'(x^-) - u'(x^+) \right], & s \in (0.5, 1), \end{cases} \tag{7}$$

where $u(x^-)$ and $u(x^+)$ denote the left and right limits of u at x respectively. Actually, $\mathcal{K}_s^* u$ can be regarded as an extension of the definition (2) if $u(x)$ has weaker regularity; based on (2) and (6), it is easy to verify that

$$\mathcal{K}_s u(x) = \mathcal{K}_s^* u(x)$$

for $u \in C[a, b]$ if $s \in [0, 0.5)$ and for $u \in C^1[a, b]$ if $s \in [0.5, 1)$.

2.1. Explicit expression of $\alpha_{ij}^{(s)}$

According to (6) with (7), we can obtain explicit expressions for $\alpha_{ij}^{(s)}$, $i = 1, 2, \ldots, n$ and $j = 0, 1, \ldots, n + 1$ by straightforward computations. If $s = 0$, we have

$$\alpha_{i,j}^{(0)} = \begin{cases} \dfrac{|x_{j-1} - x_i|}{h_j} \ln \left| \dfrac{x_{j-1} - x_i}{x_j - x_i} \right| + \dfrac{|x_{j+1} - x_i|}{h_{j+1}} \ln \left| \dfrac{x_{j+1} - x_i}{x_j - x_i} \right|, & |j - i| \ge 2, \\ \dfrac{x_{i+2} - x_i}{h_{i+2}} \ln \dfrac{x_{i+2} - x_i}{x_{i+1} - x_i}, & j = i + 1, \\ -2 + \ln h_i + \ln h_{i+1}, & j = i, \\ \dfrac{x_i - x_{i-2}}{h_{i-1}} \ln \dfrac{x_i - x_{i-2}}{x_i - x_{i-1}}, & j = i - 1 \end{cases}$$

for $1 \le i, j \le n$ and

$$\alpha_{i,0}^{(0)} = \begin{cases} 1, & i = 1, \\ 1 - \dfrac{x_i - x_1}{h_1} \ln \left| \dfrac{x_i - x_0}{x_i - x_1} \right|, & 2 \le i \le n, \end{cases}$$

$$\alpha_{i,n+1}^{(0)} = \begin{cases} 1, & i = n, \\ 1 - \dfrac{x_n - x_i}{h_{n+1}} \ln \left| \dfrac{x_{n+1} - x_i}{x_n - x_i} \right|, & 1 \le i \le n - 1. \end{cases}$$

If $s = 0.5$, then

$$\alpha_{i,j}^{(1/2)} = \begin{cases} h_j^{-1} \ln \left| \dfrac{x_j - x_i}{x_{j-1} - x_i} \right| + h_{j+1}^{-1} \ln \left| \dfrac{x_j - x_i}{x_{j+1} - x_i} \right|, & |j - i| \geq 2, \\ h_{i+1}^{-1} \ln h_{i+1} + h_{i+1}^{-1} - h_{i+2}^{-1} \ln \dfrac{x_{i+2} - x_i}{h_{i+1}}, & j = i + 1, \\ -h_i^{-1}(1 + \ln h_i) - h_{i+1}^{-1}(1 + \ln h_{i+1}), & j = i, \\ h_i^{-1} \ln h_i + h_i^{-1} - h_{i-1}^{-1} \ln \dfrac{x_i - x_{i-2}}{h_i}, & j = i - 1 \end{cases}$$

for $1 \leq i, j \leq n$ and

$$\alpha_{i,0}^{(1/2)} = \begin{cases} h_1^{-1} \ln h_1, & i = 1, \\ h_1^{-1} \ln \dfrac{x_i - x_0}{x_i - x_1} - (x_i - x_0)^{-1}, & 2 \leq i \leq n, \end{cases}$$

$$\alpha_{i,n+1}^{(1/2)} = \begin{cases} h_{n+1}^{-1} \ln h_{n+1}, & i = n, \\ h_{n+1}^{-1} \ln \dfrac{x_{n+1} - x_i}{x_n - x_i} - (x_{n+1} - x_i)^{-1}, & 1 \leq i \leq n - 1. \end{cases}$$

If $s \in (0, 1)\backslash\{0.5\}$, then

$$\alpha_{i,j}^{(s)} = \begin{cases} \dfrac{|x_j - x_i|^{1-2s} - |x_{j-1} - x_i|^{1-2s}}{(1 - 2s)2sh_j} + \dfrac{|x_j - x_i|^{1-2s} - |x_{j+1} - x_i|^{1-2s}}{(1 - 2s)2sh_{j+1}}, & |j - i| \geq 2, \\ \dfrac{h_{i+1}^{-2s} + h_{i+2}^{-1} h_{i+1}^{1-2s} - h_{i+2}^{-1}(x_{i+2} - x_i)^{1-2s}}{(1 - 2s)2s}, & j = i + 1, \\ -\dfrac{h_i^{-2s} + h_{i+1}^{-2s}}{(1 - 2s)2s}, & j = i, \\ \dfrac{h_i^{-2s} + h_{i-1}^{-1} h_i^{1-2s} - h_{i-1}^{-1}(x_i - x_{i-2})^{1-2s}}{(1 - 2s)2s}, & j = i - 1 \end{cases}$$

for $1 \leq i, j \leq n$ and

$$\alpha_{i,0}^{(s)} = \begin{cases} \dfrac{h_1^{-2s}}{1 - 2s}, & i = 1, \\ \dfrac{(x_i - x_0)^{1-2s} - (x_i - x_1)^{1-2s}}{(1 - 2s)2sh_1} - \dfrac{(x_i - x_0)^{-2s}}{2s}, & 2 \leq i \leq n, \end{cases}$$

$$\alpha_{i,n+1}^{(s)} = \begin{cases} \dfrac{h_{n+1}^{-2s}}{1 - 2s}, & i = n, \\ \dfrac{(x_{n+1} - x_i)^{1-2s} - (x_n - x_i)^{1-2s}}{(1 - 2s)2sh_{n+1}} - \dfrac{(x_{n+1} - x_i)^{-2s}}{2s}, & 1 \leq i \leq n - 1. \end{cases}$$

In particular, when a uniform grid (i.e., $h_i \equiv h$) is used, the expressions of $\alpha_{ij}^{(s)}$ can further be simplified as

$$\alpha_{i,i+m}^{(0)} = \begin{cases} 2 \ln 2, & |m| = 1, \\ -2 + 2 \ln h, & m = 0, \\ |m - 1| \ln \left| \dfrac{m - 1}{m} \right| + |m + 1| \ln \left| \dfrac{m + 1}{m} \right|, & |m| \geq 2, \end{cases}$$

$$\alpha_{i,0}^{(0)} = \begin{cases} 1, & i = 1, \\ 1 - (i - 1) \ln \left| \dfrac{i}{i - 1} \right|, & 2 \leq i \leq n, \end{cases}$$

$$\alpha_{i,n+1}^{(0)} = \begin{cases} 1, & i = n, \\ 1 - (n - i) \ln \left| \dfrac{n + 1 - i}{n - i} \right|, & 1 \leq n \leq n - 1, \end{cases}$$

if $s = 0$,

$$\alpha^{(1/2)}_{i,i+m} = \begin{cases} h^{-1}(1 - \ln 2 + \ln h), & |m| = 1, \\ -2h^{-1}(1 + \ln h), & m = 0, \\ h^{-1} \ln \dfrac{m^2}{m^2 - 1}, & |m| \geq 2, \end{cases}$$

$$\alpha^{(1/2)}_{i,0} = \begin{cases} h^{-1} \ln h, & i = 1, \\ h^{-1} \left(\ln \dfrac{i}{i-1} - \dfrac{1}{i} \right), & 2 \leq i \leq n, \end{cases}$$

$$\alpha^{(1/2)}_{i,n+1} = \begin{cases} h^{-1} \ln h, & i = n, \\ h^{-1} \left(\ln \dfrac{n+1-i}{n-i} - \dfrac{1}{n+1-i} \right), & 1 \leq i \leq n-1, \end{cases}$$

if $s = 0.5$, and

$$\alpha^{(s)}_{i,i+m} = \begin{cases} \dfrac{h^{-2s}}{(1-2s)s}(1 - 2^{-2s}), & |m| = 1, \\ -\dfrac{h^{-2s}}{(1-2s)s}, & m = 0, \\ \dfrac{h^{-2s}}{(1-2s)2s}(2|m|^{1-2s} - (|m|-1)^{1-2s} - (|m|+1)^{1-2s}), & |m| \geq 2, \end{cases}$$

$$\alpha^{(s)}_{i,0} = \begin{cases} \dfrac{h^{-2s}}{1-2s}, & i = 1, \\ \left(\dfrac{i^{1-2s} - (i-1)^{1-2s}}{2s(1-2s)} - \dfrac{i^{-2s}}{2s} \right) h^{-2s}, & 2 \leq i \leq n, \end{cases}$$

$$\alpha^{(s)}_{i,n+1} = \begin{cases} \dfrac{h^{-2s}}{1-2s}, & i = n, \\ \left(\dfrac{(n-i+1)^{1-2s} - (n-i)^{1-2s}}{(1-2s)2s} - \dfrac{(n+1-i)^{-2s}}{2s} \right) h^{-2s}, & 1 \leq i \leq n-1 \end{cases}$$

if $s \in (0, 1)\backslash\{0.5\}$.

2.2. Error analysis of the quadrature rule

In this section, we derive error estimates for the nodal-type quadrature rule for $s \in [0, 1)$.

Theorem 1. *Assume $u \in C^2(a, b)$ and let $\mathcal{Q}^{(s)}_h u(x_i)$ be defined in (4). We then have, for $i = 1, 2, \ldots, n$,*

$$\mathcal{E}(x_i; u) = |\mathcal{K}_s u(x_i) - \mathcal{Q}^{(s)}_h u(x_i)| \leq \begin{cases} Ch^2 \ln h, & s = 0, \\ Ch \ln h, & s = 0.5, \\ Ch^{2-2s}, & s \in (0, 1)\backslash\{0.5\}, \end{cases} \tag{8}$$

where $h = \max_{1 \leq i \leq n+1} h_i$.

Proof. Define $e(x) = u(x) - \Pi_h u(x)$; then

$$\begin{aligned} \mathcal{K}_s u(x_i) - \mathcal{Q}^{(s)}_h u(x_i) &= \mathcal{K}^*_s e(x_i) \\ &= \int_a^{x_{i-1}} \frac{e(x)}{(x - x_i)^{1+2s}}\, dx + \int_{x_{i+1}}^b \frac{e(x)}{(x - x_i)^{1+2s}}\, dx + ⨎_{x_{i-1}}^{x_{i+1}} \frac{u(x) - \Pi_h u(x)}{|x - x_i|^{1+2s}}\, dx, \end{aligned} \tag{9}$$

because $u \in C^2(a,b)$. By definition (6), we get

$$\fint_{x_{i-1}}^{x_{i+1}} \frac{\Pi_h u(x)}{|x-x_i|^{1+2s}}\,dx = \begin{cases} -u(x_i)\left(\frac{\ln h_i}{h_i}+\frac{\ln h_{i+1}}{h_{i+1}}\right)+u(x_{i-1})\frac{\ln h_i}{h_i}+u(x_{i+1})\frac{\ln h_{i+1}}{h_{i+1}} & \\ \quad +u(x_i)\int_{x_{i-1}}^{x_{i+1}}\frac{1}{(x-x_i)^2}\,dx, & s=0.5, \\ -u(x_i)\frac{h_i^{-2s}+h_{i+1}^{-2s}}{1-2s}+u(x_{i-1})\frac{h_i^{-2s}}{1-2s}+u(x_{i+1})\frac{h_{i+1}^{-2s}}{1-2s} & \\ \quad +u(x_i)\int_{x_{i-1}}^{x_{i+1}}\frac{1}{|x-x_i|^{1+2s}}\,dx, & \text{otherwise}. \end{cases} \tag{10}$$

Let

$$\begin{aligned} \mathcal{I}_1 &= \int_a^{x_{i-1}} \frac{e(x)}{(x_i-x)^{1+2s}}\,dx, \\ \mathcal{I}_2 &= \int_{x_{i+1}}^{b} \frac{e(x)}{(x-x_i)^{1+2s}}\,dx, \\ \mathcal{I}_3 &= \int_{x_{i-1}}^{x_{i+1}} \frac{u(x)-u(x_i)-u'(x_i)(x-x_i)}{|x-x_i|^{1+2s}}\,dx. \end{aligned} \tag{11}$$

We then obtain from (9) and (10) that

$$\mathcal{K}_s^* e(x_i) = \mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3+\begin{cases} \ln h_i\left(\frac{u(x_i)-u(x_{i-1})}{h_i}-u'(x_i)\right) & \\ \quad +\ln h_{i+1}\left(\frac{u(x_i)-u(x_{i+1})}{h_{i+1}}+u'(x_i)\right), & s=0.5, \\ \frac{h_i^{1-2s}}{1-2s}\left(\frac{u(x_i)-u(x_{i-1})}{h_i}-u'(x_i)\right) & \\ \quad +\frac{h_{i+1}^{1-2s}}{1-2s}\left(\frac{u(x_i)-u(x_{i+1})}{h_{i+1}}+u'(x_i)\right), & \text{otherwise}, \end{cases} \tag{12}$$

where we use the fact

$$\int_{x_{i-1}}^{x_{i+1}} \frac{x-x_i}{|x-x_i|^{1+2s}}\,dx = \begin{cases} \frac{h_{i+1}^{1-2s}-h_i^{1-2s}}{1-2s}, & s\in[0,1)\backslash\{0.5\}, \\ \ln\frac{h_{i+1}}{h_i}, & s=0.5. \end{cases}$$

By noting that $u(x) \in C^2[a,b]$ and using the property of Lagrange interpolation, we have

$$\left|\frac{u(x_i)-u(x_{i-1})}{h_i}-u'(x_i)\right| \le Ch_i, \qquad \left|\frac{u(x_i)-u(x_{i+1})}{h_{i+1}}+u'(x_i)\right| \le Ch_{i+1},$$

and

$$|e(x)| \le \max_{x\in[x_{i-1},x_i]}\left|\frac{1}{2}(x-x_i)(x-x_{i+1})u''(\xi_x)\right| \le Ch^2, \tag{13}$$

where $\xi_x \in (x_i, x_{i+1})$. Thus, for $i = 1, 2, 3$, we have

$$|\mathcal{I}_i| \le \begin{cases} Ch^2\ln h, & s=0, \\ Ch^{2-2s}, & s\in(0,1). \end{cases} \tag{14}$$

Then (8) follows immediately from (12) and (14) and the proof is complete. □

2.3. Modified quadrature rule for stronger singularity

A well-known second-order approximation of $u'(x_i)$ valid for general grids is given by

$$u'(x_i) = -\frac{h_{i+1}}{h_i(h_i+h_{i+1})}u(x_{i-1}) + \frac{h_{i+1}^2-h_i^2}{h_ih_{i+1}(h_i+h_{i+1})}u(x_i) + \frac{h_i}{h_{i+1}(h_i+h_{i+1})}u(x_{i+1}) + O(h^2).$$

Substituting into (12), we get

$$\mathcal{K}_s^* e(x_i) = \left(\sum_{k=1}^{3}\mathcal{I}_k\right) + \left(\frac{u(x_i)}{h_ih_{i+1}} - \frac{u(x_{i-1})}{h_i(h_i+h_{i+1})} - \frac{u(x_{i+1})}{(h_i+h_{i+1})h_{i+1}}\right)\eta_i^{(s)} + \begin{cases} O(h^2\ln h), & s=0.5, \\ O(h^{3-2s}), & s\in[0,1)\backslash\{0.5\}, \end{cases} \tag{15}$$

where

$$\eta_i^{(s)} = \begin{cases} h_i\ln h_i + h_{i+1}\ln h_{i+1}, & s=0.5, \\ \dfrac{h_i^{2-2s}+h_{i+1}^{2-2s}}{1-2s}, & s\in[0,1)\backslash\{0.5\}. \end{cases}$$

Therefore, we propose a modified quadrature rule that can improve the stability of the collocation schemes proposed in Sections 3 and 4 for $s\in[0.5,1)$ as follows:

$$\begin{aligned}\tilde{\mathcal{Q}}_h^{(s)}u(x_i) &= \mathcal{Q}_h^{(s)}u(x_i) + \left(\frac{u(x_i)}{h_ih_{i+1}} - \frac{u(x_{i-1})}{(h_i+h_{i+1})h_i} - \frac{u(x_{i+1})}{(h_i+h_{i+1})h_{i+1}}\right)\eta_i^{(s)} \\ &= \sum_{j=0}^{n+1}\tilde{\alpha}_{ij}^{(s)}u(x_j)\end{aligned} \tag{16}$$

for $1\le i\le n$, where

$$\tilde{\alpha}_{i,j}^{(s)} = \begin{cases} \alpha_{i,j}^{(s)} - \dfrac{\eta_i^{(s)}}{h_i(h_i+h_{i+1})}, & j=i-1, \\ \alpha_{i,j}^{(s)} + \dfrac{\eta_i^{(s)}}{h_ih_{i+1}}, & j=i, \\ \alpha_{i,j}^{(s)} - \dfrac{\eta_i^{(s)}}{h_{i+1}(h_i+h_{i+1})}, & j=i+1, \\ \alpha_{i,j}^{(s)}, & |j-i|\ge 2, \end{cases}$$

$$\tilde{\alpha}_{i,0}^{(s)} = \begin{cases} \alpha_{1,0}^{(s)} - \dfrac{\eta_1^{(s)}}{h_1(h_1+h_2)}, & i=1, \\ \alpha_{i,0}^{(s)}, & 2\le i\le n, \end{cases}$$

$$\tilde{\alpha}_{i,n+1}^{(s)} = \begin{cases} \alpha_{n,n+1}^{(s)} - \dfrac{\eta_n^{(s)}}{h_{n+1}(h_n+h_{n+1})}, & i=n, \\ \alpha_{i,n+1}^{(s)}, & 1\le i\le n-1. \end{cases}$$

We next provide an estimate for the error of the quadrature rule $\tilde{\mathcal{Q}}_h^{(s)}u(x_i)$, whose proof can be obtained in the same way as shown above.

Theorem 2. *Assume $u\in C^2(a,b)$ and let $\tilde{\mathcal{Q}}_h^{(s)}u(x_i)$ be defined in (16). Then, we have*

$$\mathcal{E}(x_i;u) = |\mathcal{K}_s u(x_i) - \tilde{\mathcal{Q}}_h^{(s)}u(x_i)| \le \begin{cases} Ch^2\ln h, & s=0, \\ Ch, & s=0.5, \\ Ch^{2-2s}, & s\in(0,1)\backslash\{0.5\} \end{cases} \tag{17}$$

for $i=1,2,\ldots,n$, where $h=\max_{i=1,\ldots,n+1}h_i$.

Remark 1. If the grid is uniform, $\tilde{\mathcal{Q}}_h^{(s)} u(x_i)$ can be simplified to

$$\begin{aligned}\tilde{\mathcal{Q}}_h^{(s)} u(x_i) &= \mathcal{Q}_h^{(s)} u(x_i) + [2u(x_i) - u(x_{i-1}) - u(x_{i+1})]\theta_s \\ &= \sum_{j=0}^{n+1} \tilde{\alpha}_{ij}^{(s)} u(x_j), \quad 1 \le i \le n,\end{aligned} \tag{18}$$

where

$$\tilde{\alpha}_{i,i+m}^{(s)} = \begin{cases} \alpha_{i,i+m}^{(s)} - \theta_s, & |m| = 1, \\ \alpha_{i,i+m}^{(s)} + 2\theta_s, & m = 0, \\ \alpha_{i,i+m}^{(s)}, & |m| \ge 2, \end{cases}$$

with

$$\theta_s = \begin{cases} h^{-1} \ln h, & s = 0.5, \\ \dfrac{h^{-2s}}{1-2s}, & 0.5 < s < 1, \end{cases}$$

and

$$\tilde{\alpha}_{i,0}^{(s)} = \begin{cases} 0, & i = 1, \\ \alpha_{i,0}^{(s)}, & 2 \le i \le n, \end{cases} \qquad \tilde{\alpha}_{i,n+1}^{(s)} = \begin{cases} 0, & i = n, \\ \alpha_{i,n+1}^{(s)}, & 1 \le i \le n-1. \end{cases}$$

3. Collocation method for the finite-part integral equation

In this section, we investigate a linear collocation scheme for solving the following finite-part integral equation:

$$\begin{cases} \displaystyle\bar{\int_a^b} \frac{u(y)}{|x-y|^{1+2s}}\, dy = f(x), & x \in (a,b), \\ u(a) = u_a, & u(b) = u_b. \end{cases} \tag{19}$$

Let us apply the quadrature rule $\mathcal{Q}_h^{(s)} u$ in (4) if $s \in [0, 0.5)$ and $\tilde{\mathcal{Q}}_h^{(s)} u$ in (16) if $s \in [0.5, 1)$ to the integral on the left-hand side of (19), and collocate the equation at the nodal points x_i, $i = 1, 2, \ldots, n$. We then get a linear collocation scheme whose resulting matrix form is

$$\mathbb{A}\vec{U} = \vec{F}, \tag{20}$$

where

$$\vec{U} = (u_1, u_2, \ldots, u_n)^T$$

and

$$\mathbb{A} = (a_{ij}^{(s)}) \in \mathbb{R}^{n\times n}, \qquad \vec{F} = (f_1, f_2, \ldots, f_n)^T$$

with

$$f_i = f(x_i) - a_{i,0}^{(s)} u(a) - a_{i,n+1}^{(s)} u(b) \tag{21}$$

and

$$a_{ij}^{(s)} = \begin{cases} \alpha_{ij}^{(s)}, & s \in [0, 0.5), \\ \tilde{\alpha}_{ij}^{(s)}, & s \in [0.5, 1) \end{cases} \tag{22}$$

for $i = 1, \ldots, n$ and $j = 0, \ldots, n+1$. The collocation scheme (20) is valid on general grids although below we only present the error analysis for uniform grids.

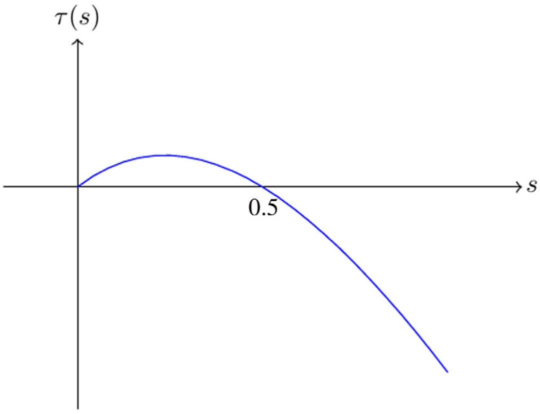


Fig. 1. Plot of $\tau(s) = 1 - s - 2^{-2s}$.

Lemma 3. *Assume that the partition π_h is uniform and $\mathbb{A}$ is the matrix defined in the collocation scheme (20). Then*
(1) *$\mathbb{A}$ is a Toeplitz matrix;*
(2) *$-\mathbb{A}$ is a symmetric strictly diagonally dominant M-matrix;*
(3) *the linear equation (20) has a unique solution.*

Proof. From (22) and the explicit expression of $\alpha_{ij}^{(s)}$ and $\tilde{\alpha}_{ij}^{(s)}$, we can easily find that $\mathbb{A}$ is a Toeplitz matrix. We proceed with the proof for the case $s \in (0, 0.5) \cup (0.5, 1)$; the other cases similarly follow. Let us first check the sign symbol of the diagonal elements of $\mathbb{A}$; it is easy to see that

$$a_{ii}^{(s)} = \begin{cases} \alpha_{ii}^{(s)} = -\dfrac{h^{-2s}}{(1-2s)s} < 0, & s \in (0, 0.5) \\ \tilde{\alpha}_{ii}^{(s)} = -\dfrac{h^{-2s}}{s} < 0, & s \in (0.5, 1). \end{cases} \tag{23}$$

The sign symbol of the first off-diagonal elements of $\mathbb{A}$ can be identified as

$$a_{i,i\pm1}^{(s)} = \begin{cases} \alpha_{i,i\pm1}^{(s)} = \dfrac{h^{-2s}}{(1-2s)s}(1 - 2^{-2s}) > 0, & s \in (0, 0.5), \\ \tilde{\alpha}_{i,i\pm1}^{(s)} = \dfrac{h^{-2s}(1 - s - 2^{-2s})}{(1-2s)s} > 0, & s \in (0.5, 1), \end{cases} \tag{24}$$

where the sign symbol of the function $\tau(s) = 1 - s - 2^{-2s}$ can be determined from Fig. 1.

For the other entries, we observe from (5) that they are just classic Riemann integrals for $m \geq 2$, which are positive due to the positiveness of the integrands. Moreover, for $i = 2, \ldots, n-1$,

$$\sum_{j=1}^{n} a_{ij}^{(s)} = -\frac{(x_i - a)^{1-2s} - (x_{i-1} - a)^{1-2s} + (b - x_i)^{1-2s} - (b - x_{i+1})^{1-2s}}{h(1-2s)2s}. \tag{25}$$

To determine the sign symbol of (25), let us introduce a function $\kappa(x) = x^{1-2s}$. It is easy to verify that for $x > 0$, $\kappa(x)$ is monotonically increasing if $s \in (0, 0.5)$ and monotonically decreasing if $s \in (0.5, 1)$, and hence

$$\frac{x^{1-2s} - (x-h)^{1-2s}}{1-2s} > 0, \quad s \in (0, 1).$$

Thus we obtain

$$\sum_{j=1}^{n} a_{ij}^{(s)} < 0.$$

Putting all of these together leads to a fact that $-\mathbb{A}$ is an M-matrix. The strict diagonal dominance of $\mathbb{A}$ and the uniqueness of (20) follows immediately. $\square$

Remark 2. The multiplication of a vector by an $n \times n$ Toeplitz matrix can be performed efficiently by a fast Fourier transform (FFT) with computational cost of $O(n \ln n)$ which is much lower than that for the usual matrix–vector multiplication. Thus, we can apply an FFT for the solution of the linear system (20) by iterative solvers.

Before stating a crucial lemma used in the proof of our main result, we introduce an important result given in [19]: if $\mathbb{A} = (a_{ij}) \in \mathbb{R}^{n\times n}$ is a strictly diagonally dominant matrix, then

$$\|\mathbb{A}^{-1}\|_\infty \le \max_{1\le i\le n} \frac{1}{|a_{ii}| - \sum\limits_{j\neq i} |a_{ij}|}, \quad 1 = 1, 2, \ldots, n. \tag{26}$$

Lemma 4. *Assume that the partition π_h is uniform and $\mathbb{A}$ is the matrix defined in the collocation scheme (20). Then, for sufficiently small grid size $h > 0$, we have*

$$\|\mathbb{A}^{-1}\|_\infty \le \begin{cases} \left|2 \ln \dfrac{b-a}{2}\right|^{-1}, & s = 0, \\ \left(\dfrac{b-a}{2}\right)^{2s} s, & s \in (0, 1). \end{cases} \tag{27}$$

Proof. Without loss of generality, we only consider the case $s \in (0, 0.5) \cup (0.5, 1)$. Note first $-\mathbb{A}$ is a M-matrix as shown in Lemma 3. Next, it is easy to see from (25) that the minimum value of $|a_{ii}^{(s)}| - \sum_{j\neq i} |a_{ij}^{(s)}|$ is achieved at the mid-point of (a, b) so that

$$|a_{ii}^{(s)}| - \sum_{j\neq i} |a_{ij}^{(s)}| \quad \ge \quad \frac{(\frac{b-a}{2})^{1-2s} - (\frac{b-a}{2} - h)^{1-2s}}{h(1-2s)s} = \frac{(\frac{b-a}{2})^{-2s}}{s} + O(h).$$

Then, (27) immediately follows by using (26). $\square$

Theorem 5. *Assume that the partition π_h is uniform and $\mathbb{A}$ is the matrix defined in the collocation scheme (20). Then,*

$$\mathrm{Cond}_\infty(\mathbb{A}) \le \begin{cases} C \ln n, & s = 0, \\ C n^{2s}, & s \in (0, 1). \end{cases} \tag{28}$$

Proof. From the proof of Lemma 3, we see that $a_{ii}^{(s)} \langle 0, a_{ij}^{(s)} \rangle 0 (j \neq i)$ and $\sum_{j=1}^n a_{ij}^{(s)} < 0$. Then for $i = 1, 2, \ldots, n$,

$$\|\mathbb{A}\|_\infty = \sum_{j=1}^n |a_{ij}^{(s)}| = -2a_{ii} + \sum_{j=1}^n a_{ij}^{(s)} < -2a_{ii}.$$

From the expression of a_{ii} in (22) we have that $\|\mathbb{A}\|_\infty \le C \ln n$ if $s = 0$ and $\|\mathbb{A}\|_\infty \le C n^{2s}$ if $s \in (0, 1)$. Finally, (28) immediately follows by using (27). $\square$

We define a discrete maximum norm by

$$\|u\|_\infty := \max_{1\le i\le n} |u(x_i)|. \tag{29}$$

We conclude this section with the main result.

Theorem 6. *Suppose that the partition π_h is uniform and $u(x)$, the solution of Eq. (19), belongs to $C^2(a, b)$. Then, it holds that for sufficiently small grid size $h > 0$,*

$$\|u - u^h\|_\infty \le \begin{cases} C h^2 \ln h, & s = 0, \\ C \left(\dfrac{b-a}{2}\right)^{2s} s h^{2-2s}, & s \in (0, 1), \end{cases} \tag{30}$$

where $u^h(x) = \sum_{j=1}^n u_j \phi_j(x)$ is the approximate solution computed from the linear system (20).

Proof. Let $\vec{U}_e = [u(x_1)\ \cdots\ u(x_n)]^T$ denote the exact solution vector. From Lemma 3, we see that $\mathbb{A}$ is invertible and thus

$$\vec{U}_e - \vec{U} = \mathbb{A}^{-1}(\mathbb{A}\vec{U}_e - \vec{F})$$

which implies

$$u(x_i) - u_i = \sum_{1\le k\le n} b_{ik}\mathcal{E}(x_i; u),$$

where $\mathcal{E}(x_i; u)$ is defined in (8) for $0 \le s < 0.5$ and (17) for $0.5 \le s < 1$. Then, by Theorems 1, 2 and Lemma 4, we have for $s \in (0, 1)$

$$|u(x_i) - u_i| \le Ch^{2-2s} \sum_{1\le k\le n} |b_{ik}| \le Ch^{2-2s}\|\mathbb{A}^{-1}\|_\infty \le C\left(\frac{b-a}{2}\right)^{2s} sh^{2-2s}.$$

The result for $s = 0$ can be derived in a similar manner. □

4. Collocation method for the nonlocal diffusion problem

In this section, we study a linear steady-state nonlocal reaction–diffusion model given by the integral equation

$$-\mathcal{L}_{\delta,s}u(x) + \lambda(x)u(x) = g(x), \quad x \in (a, b), \tag{31}$$

where $u(x)$ denotes the displacement of the material,

$$\mathcal{L}_{\delta,s}u(x) = \int_{a-\delta}^{b+\delta} \frac{u(y) - u(x)}{|x-y|^{1+2s}}\,dy \tag{32}$$

is the nonlocal diffusion operator with $s \ge 0$ being a parameter and $\delta > 0$ being called the *horizon* that characterizes the extent of the influence of the kernel function, $\lambda(x) \ge 0$ is the reaction coefficient, and $g(x)$ represents the prescribed forcing term. The volume constraint

$$u(x) = u_a(x), \quad x \in (a-\delta, a) \quad \text{and} \quad u(x) = u_b(x), \quad x \in (b, b+\delta) \tag{33}$$

is imposed.

Using similar approaches as those in the preceding section, we can propose a collocation-type discretization for (31) as follows: for $i = 1, 2, \ldots, n$,

$$(\lambda_i + \tau_i^s)u(x_i) - \hat{\mathcal{Q}}_h^{(s)}u(x_i) \approx g^{(s)}(x_i) + \int_{a-\delta}^{a} \frac{u_a(y)}{|x_i-y|^{1+2s}}\,dy + \int_{b}^{b+\delta} \frac{u_b(y)}{|x_i-y|^{1+2s}}\,dy, \tag{34}$$

where $\lambda_i = \lambda(x_i)$, $\hat{\mathcal{Q}}_h^{(s)}u(x_i)$ is the quadrature rule as

$$\hat{\mathcal{Q}}_h^{(s)}u(x_i) = \begin{cases} \mathcal{Q}_h^{(s)}u(x_i), & s \in [0, 0.5), \\ \tilde{\mathcal{Q}}_h^{(s)}u(x_i), & s \in [0.5, 1) \end{cases}$$

with $\mathcal{Q}_h^{(s)}u$ and $\tilde{\mathcal{Q}}_h^{(s)}u$ being defined in (4) and (16), respectively, and

$$\begin{aligned} \tau_i^s &= \fint_{a-\delta}^{b+\delta} \frac{1}{|x_i-y|^{1+2s}}\,dy \\ &= \begin{cases} \ln|(x_i - a + \delta)(b + \delta - x_i)|, & s = 0, \\ -\dfrac{(x_i - a + \delta)^{-2s} + (b + \delta - x_i)^{-2s}}{2s}, & s \in (0, 1). \end{cases} \end{aligned} \tag{35}$$

Set the entries of the $n \times n$ matrix $\mathbb{C}$ to be

$$c_{ii}^{(s)} = \lambda_i + \tau_i^s - \begin{cases} \alpha_{ii}^{(s)}, & s \in [0, 0.5), \\ \tilde{\alpha}_{ii}^{(s)}, & s \in [0.5, 1) \end{cases} \tag{36}$$

for $i = 1, \ldots, n$, and

$$c_{ij}^{(s)} = \begin{cases} -\alpha_{ij}^{(s)}, & s \in [0, 0.5) \\ -\tilde{\alpha}_{ij}^{(s)}, & s \in [0.5, 1) \end{cases} \tag{37}$$

for $i, j = 1, \ldots, n$ and $i \neq j$. Let the entries of the right-hand side n-vector $\vec{F}$ be defined by

$$f_i = g(x_i) - c_{i,0}^{(s)} u(a) - c_{i,n+1}^{(s)} u(b) + \int_{a-\delta}^{a} \frac{u_a(y)}{|x_i - y|^{1+2s}}\, dy + \int_{b}^{b+\delta} \frac{u_b(y)}{|x_i - y|^{1+2s}}\, dy \tag{38}$$

for $i = 1, \ldots, n$. Then, a collocation approximation of the nonlocal reaction–diffusion problem (31) can be expressed by the linear system

$$\mathbb{C}\vec{U} = \vec{F}. \tag{39}$$

Lemma 7. *Assume that the partition π_h is uniform and $\mathbb{C}$ is the matrix defined in the collocation scheme (39). Then,*

(1) *$\mathbb{C}$ is a sum of a diagonal matrix and a Toeplitz matrix;*
(2) *$\mathbb{C}$ is a symmetric strictly diagonally dominant M-matrix;*
(3) *the linear system (39) has a unique solution.*

Proof. It is obvious that $\mathbb{C} = \mathrm{diag}(\lambda_i + \tau_i^s) - \mathbb{A}$ and $\mathbb{A}$ is the Toeplitz matrix defined by (20). For simplicity, we only consider the case $s \in (0, 0.5) \cup (0.5, 1)$ because the other cases can be proved in a similar way. It follows after (36) and (37) and the proof of Lemma 3 that $c_{ij}^{(s)} < 0$ if $i \neq j$. It remains to determine the sign symbol of the diagonal entries of $\mathbb{C}$. In fact,

$$c_{ii}^{(s)} = \begin{cases} \lambda_i + \dfrac{h^{-2s}}{(1-2s)s} - \dfrac{(x_i - a + \delta)^{-2s} + (b + \delta - x_i)^{-2s}}{2s} > 0, & s \in (0, 0.5), \\ \lambda_i + \dfrac{h^{-2s}}{s} - \dfrac{(x_i - a + \delta)^{-2s} + (b + \delta - x_i)^{-2s}}{2s} > 0, & s \in (0.5, 1). \end{cases} \tag{40}$$

To show the strictly diagonal dominance of $\mathbb{C}$, let us still consider the function $\kappa(x) = x^{1-2s}$. By Taylor expansion it holds

$$\frac{x^{1-2s} - (x-h)^{1-2s}}{h(1-2s)2s} - \frac{x^{-2s}}{2s} = \frac{1}{2} h x^{-2s-1} + O(h^2) > 0$$

for sufficiently small $h > 0$. Consequently, we get

$$\begin{aligned} \sum_{j=1}^{n} c_{ij}^{(s)} &= \lambda_i + \frac{(x_i - a)^{1-2s} - (x_{i-1} - a)^{1-2s} + (b - x_i)^{1-2s} - (b - x_{i+1})^{1-2s}}{h(1-2s)2s} \\ &\quad - \frac{(x_i - a + \delta)^{-2s} + (b + \delta - x_i)^{-2s}}{2s} \\ &> 0, \end{aligned}$$

for $i = 2, \ldots, n-1$, and thus we conclude that $\mathbb{C}$ is a M-matrix and the system (39) has a unique solution. □

Lemma 8. *Assume that the partition π_h is uniform and $\mathbb{C}$ is the matrix defined in the collocation scheme (39). Then for sufficiently small grid size $h > 0$, we have*

$$\|\mathbb{C}^{-1}\|_\infty \leq \frac{1}{\lambda_{\min} + O(\delta + h)}, \tag{41}$$

where $\lambda_{\min}$ is the lower bound of $\lambda(x)$.

Proof. Without loss of generality, we here only give the proof for the case $(0, 0.5) \cup (0.5, 1)$. Based on Lemma 7, by using Taylor expansion we have

$$\begin{aligned}|c_{ii}| - \sum_{j\neq i} |c_{ij}| &= \lambda_i + \frac{h[(x_i - a)^{-1-2s} + (b - x_i)^{-1-2s}] + O(h^2)}{2} \\ &\quad + \frac{(x_i - a)^{-2s} - (x_i - a + \delta)^{-2s} + (b - x_i)^{-2s} - (b - x_i + \delta)^{-2s}}{2s} \\ &\geq \lambda_i + O(\delta + h).\end{aligned}$$

Then, using the inequality (26), we get (41). □

We specially remark that in Lemma 8 the estimate for the upper bound of $\|\mathbb{C}^{-1}\|_\infty$ is valid for any fixed positive δ, even if $\lambda_{\min} = 0$.

Theorem 9. *Assume that the partition π_h is uniform and $\mathbb{C}$ is the matrix defined in the collocation scheme (39). Then,*

$$\mathrm{Cond}_\infty(\mathbb{C}) \leq \begin{cases} C \ln n, & s = 0, \\ Cn^{2s}, & s \in (0, 1). \end{cases} \tag{42}$$

Proof. From the proof of Lemma 7, we see that $c_{ii}^{(s)} > 0, c_{ij}^{(s)} < 0 (j \neq i)$ and $\sum_{j=1}^{n} c_{ij}^{(s)} > 0$. Then for $i = 1, 2, \ldots, n$,

$$\|\mathbb{C}\|_\infty = \sum_{j=1}^{n} |c_{ij}^{(s)}| = 2c_{ii} - \sum_{j=1}^{n} c_{ij}^{(s)} < 2c_{ii}.$$

From the explicit expression of c_{ii} in (36) we have that $\|\mathbb{A}\|_\infty \leq C \ln n$ if $s = 0$ and $\|\mathbb{A}\|_\infty \leq Cn^{2s}$ if $s \in (0, 1)$. Finally, (42) immediately follows by using (41). □

With the estimate (41), we can obtain an error estimate for the collocation scheme (39); the proof is as same as that for Theorem 6.

Theorem 10. *Suppose that the partition π_h is uniform and $u(x)$, the solution of Eq. (31), belongs to $C^2(a, b)$. Then, it holds that for fixed δ and sufficiently small grid size $h > 0$,*

$$\|u - u^h\|_\infty \leq \begin{cases} Ch^2 \ln h, & s = 0, \\ Ch^{2-2s}, & s \in (0, 1), \end{cases} \tag{43}$$

where $u^h(x) = \sum_{j=1}^{n} u_j \phi_j(x)$ is the approximate solution computed from the linear system (39).

Remark 3. Although volume constraints are still a viable constraint on solutions, for $s \in (0.5, 1)$ a standard boundary condition of the form

$$u(a) = u_a, \qquad u(b) = u_b$$

can also be specified to close the model (31) together with $\mathcal{L}_{0,s}u$ defined in (32); see [20,21]. In this case, Lemma 7 is still valid, which guarantees the solvability of the collocation scheme. The result in Lemma 8 should be modified correspondingly as

$$\|\mathbb{C}^{-1}\|_\infty \leq \frac{1}{\lambda_{\min} + O(h)}. \tag{44}$$

If $\lambda_{\min}$ is a positive constant, then $\|\mathbb{C}^{-1}\|_\infty$ is bounded and thus the result in Theorems 9 and 10 is still valid and optimal; otherwise, $\|\mathbb{C}^{-1}\|_\infty \leq O(h^{-1})$ holds, and we fail to obtain a condition number estimate as in (42) and an error estimate as in (43) for the collocation scheme.

Table 1
Example 1: Numerical results on $\|\mathbb{A}^{-1}\|_\infty$ of the collocation scheme for the finite-part integral equation (19).

h	$s=0$	$s=0.3$	$s=0.5$	$s=0.8$
$1/2^5$	0.570	0.155	0.164	0.101
$1/2^6$	0.580	0.157	0.162	0.094
$1/2^7$	0.585	0.158	0.161	0.090
$1/2^8$	0.588	0.159	0.160	0.087
$1/2^9$	0.589	0.159	0.160	0.084
$1/2^{10}$	0.590	0.159	0.159	0.083
$1/2^{11}$	0.590	0.159	0.159	0.082
$1/2^{12}$	0.591	0.159	0.159	0.081

5. Numerical experiments

We now provide some numerical examples to illustrate the performance of the proposed discretization schemes and illustrate the related error analyses. First define the quantity

$$\mu_m^{(s)}(x) = \int_0^1 \frac{(y-x)^m}{|y-x|^{1+2s}}\,dy = \begin{cases} \ln(1-x)x, & m=0 \text{ and } s=0,\\ \ln\dfrac{1-x}{x}, & m=1 \text{ and } s=0.5,\\ \dfrac{(1-x)^{m-2s}+(-1)^m x^{m-2s}}{m-2s}, & \text{otherwise.} \end{cases}$$

In the following examples, we will check the maximum nodal error norm $e_\infty = \max\limits_{1\le i\le n} |u(x_i)-u_i|$, the truncation error

$$e_{truc} = \max_{1\le i\le n} \mathcal{E}(x_i;u),$$

where $\mathcal{E}(x_i;u)(i=1,2,\dots,n)$ is defined in (8) for $s\in[0,0.5)$ and (17) for $s\in[0.5,1)$, the condition number of the stiffness matrix and the bound of its inverse. Uniform grids are used for all examples.

Example 1. We use the collocation scheme for solving the finite-part integral equation (19) in the interval [0, 1]. We use

$$u(x) = x^2(1-x)^2 \tag{45}$$

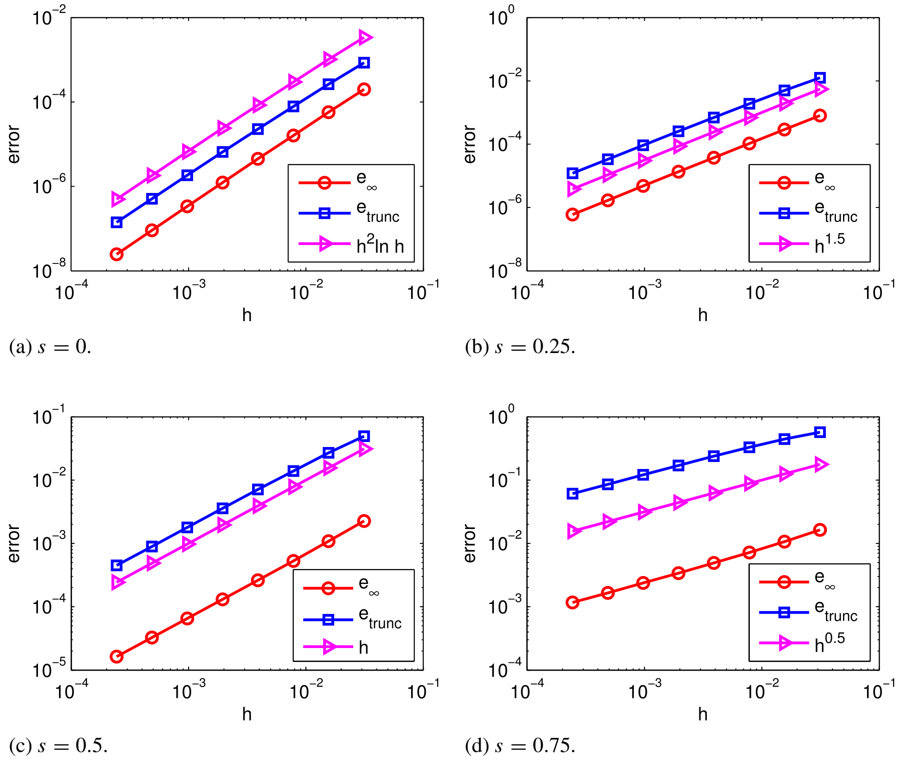
as the exact solution of (19) for which it is obvious that zero boundary conditions are satisfied. The corresponding right-hand side term is given by

$$\begin{aligned} f(x) &= x^2(1-x)^2\mu_0^{(s)}(x) + (4x^3-6x^2+2x)\mu_1^{(s)}(x)\\ &\quad + (6x^2-6x+1)\mu_2^{(s)}(x) + (4x-2)\mu_3^{(s)}(x) + \mu_4^{(s)}(x), \quad s\in[0,1). \end{aligned} \tag{46}$$

Table 1 reports the values of $\|\mathbb{A}^{-1}\|_\infty$ for several values of the parameter s and grid sizes h. We can easily see that $\|\mathbb{A}^{-1}\|_\infty$ is indeed bounded and converges to a certain number for any fixed s as the grid size h decreases. The maximum nodal errors e_∞ and truncation errors e_{trunc} are graphically presented as log–log plots of errors versus the grid size h in Fig. 2, and the condition numbers $\mathrm{Cond}_\infty(\mathbb{A})$ are graphically presented as log–log plots of condition numbers versus n in Fig. 3. It can be explicitly observed that e_∞ and e_{truc} are about $O(h^2\ln h)$ if $s=0$ and $O(h^{2-2s})$ if $s\in(0,1)$, and $\mathrm{Cond}_\infty(\mathbb{A})$ are about $O(\ln n)$ if $s=0$ and $O(n^{2s})$ if $s\in(0,1)$. All of these results are agree well with the conclusions in Theorem 1, Theorem 2, Lemma 4, Theorem 5 and Theorem 6.

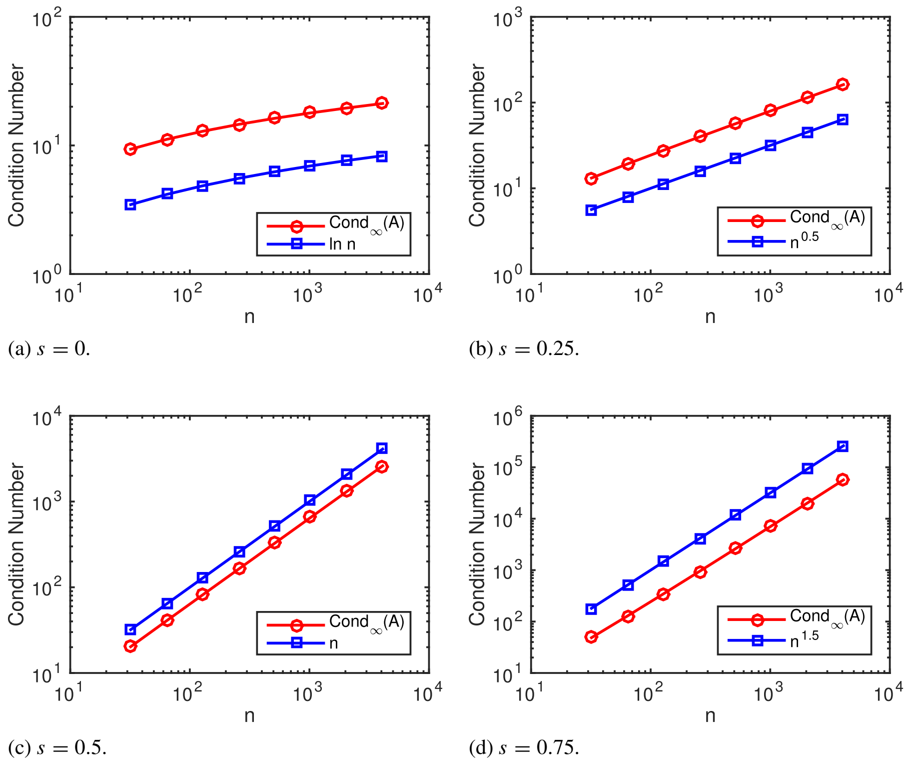
Example 2. We next test the collocation scheme for solving the nonlocal reaction–diffusion problem (31). A volume constraint boundary condition of the form

$$\begin{cases} u(x) = u_a(x), & x\in(a-\delta,a),\\ u(x)=u_b(x), & x\in(b,b+\delta) \end{cases} \tag{47}$$



(a) $s = 0$. (b) $s = 0.25$. (c) $s = 0.5$. (d) $s = 0.75$.

Fig. 2. Example 1: Maximum nodal errors and truncation errors of the collocation scheme for the finite-part integral equation (19).

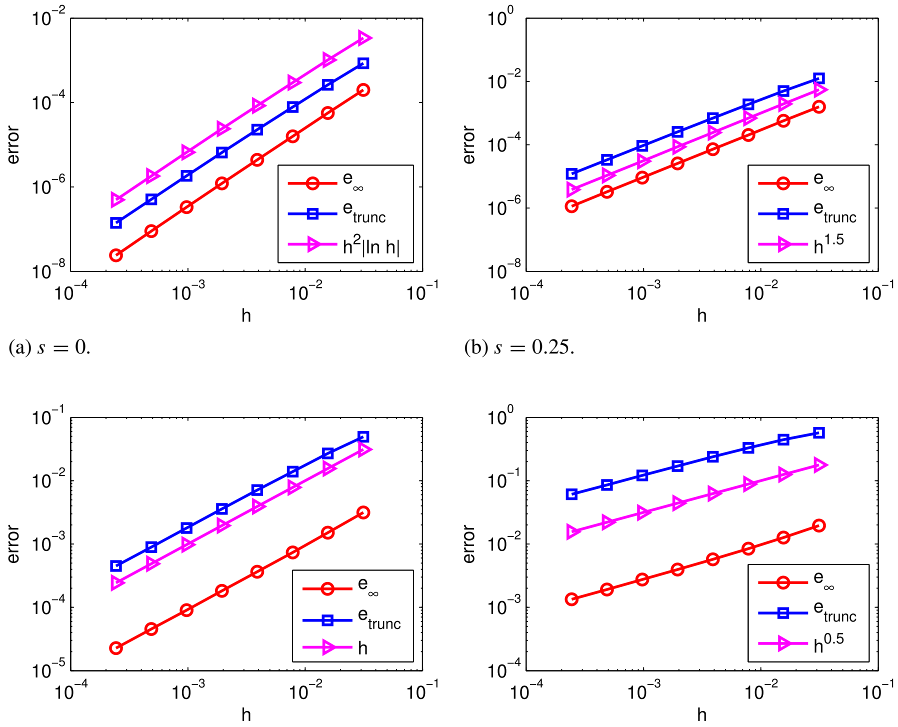


(a) $s = 0$. (b) $s = 0.25$. (c) $s = 0.5$. (d) $s = 0.75$.

Fig. 3. Example 1: Condition numbers of the stiffness matrix of the collocation scheme for the finite-part integral equation (19).

Table 2
Example 2: Numerical results on $\|\mathbb{C}^{-1}\|_\infty$ of the collocation scheme for the nonlocal diffusion problem (31) with $\lambda \equiv 1$.

h	$s = 0$	$s = 0.25$	$s = 0.5$	$s = 0.75$
$1/2^5$	0.624	0.435	0.281	0.134
$1/2^6$	0.639	0.454	0.274	0.121
$1/2^7$	0.647	0.464	0.271	0.113
$1/2^8$	0.651	0.470	0.269	0.108
$1/2^9$	0.653	0.472	0.268	0.105
$1/2^{10}$	0.654	0.473	0.267	0.102
$1/2^{11}$	0.655	0.474	0.267	0.101
$1/2^{12}$	0.655	0.474	0.266	0.100



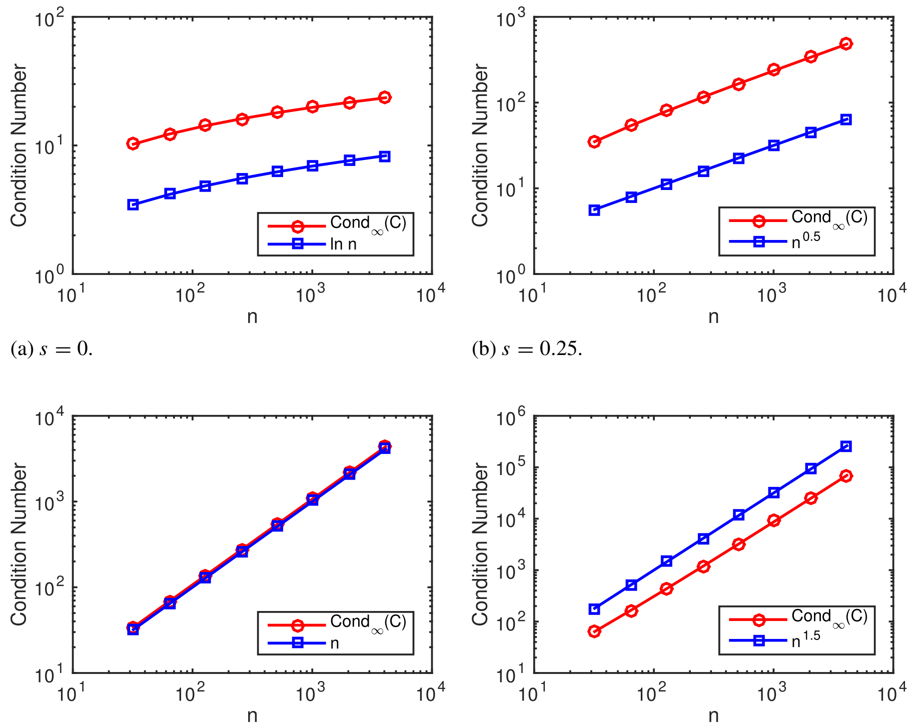
(a) $s = 0$. (b) $s = 0.25$. (c) $s = 0.5$. (d) $s = 0.75$.

Fig. 4. Example 2: Maximum nodal errors and truncation errors of the collocation scheme for the nonlocal diffusion problem (31) with $\lambda \equiv 1$.

is imposed for some $\delta > 0$. We set $a = 0$, $b = 1$, and $\delta = 0.1$, and take the same $u(x)$ defined in (45) as the exact solution. Then, the right-hand side term of (31) is determined correspondingly.

We first choose $\lambda \equiv 1$ in (31), which means that a nonzero reaction term exists. Table 2 reports the values of $\|\mathbb{C}^{-1}\|_\infty$ for some different values of s and the grid size h. It is easy to see that, for any fixed s and for decreasing h, $\|\mathbb{C}^{-1}\|_\infty$ is bounded and also stably convergent. From the log–log plots given in Fig. 4 we observe that both the maximum error e_∞ and the truncation error e_{truc} are about $O(h^2 \ln h)$ if $s = 0$ and $O(h^{2-2s})$ if $s \in (0, 1)$, which agree well with the conclusions in Lemma 8 and Theorem 10. Fig. 5 reports that the condition numbers $\mathrm{Cond}_\infty(\mathbb{C})$ are about $O(\ln^{-1} n)$ if $s = 0$ and $O(n^{2s})$ if $s \in (0, 1)$, which agree well with the conclusions in Theorem 9.

We next provide simulation results for the collocation scheme for the case of $\lambda \equiv 0$ in (31), i.e., the pure nonlocal diffusion problem. Table 3 shows the boundedness of $\|\mathbb{C}^{-1}\|_\infty$ for different s and for decreasing grid size h. Fig. 6 reports that corresponding maximum nodal errors and truncation errors are still about $O(h^2 \ln h)$ if $s = 0$ and $O(h^{2-2s})$ if $s \in (0, 1)$, which again agree well with the conclusions in Lemma 8 and Theorem 10, and Fig. 7 reports



(a) $s = 0$. (b) $s = 0.25$. (c) $s = 0.5$. (d) $s = 0.75$.

Fig. 5. Example 2: Condition numbers of the stiffness matrix of the collocation scheme for the nonlocal diffusion problem (31) with $\lambda \equiv 1$.

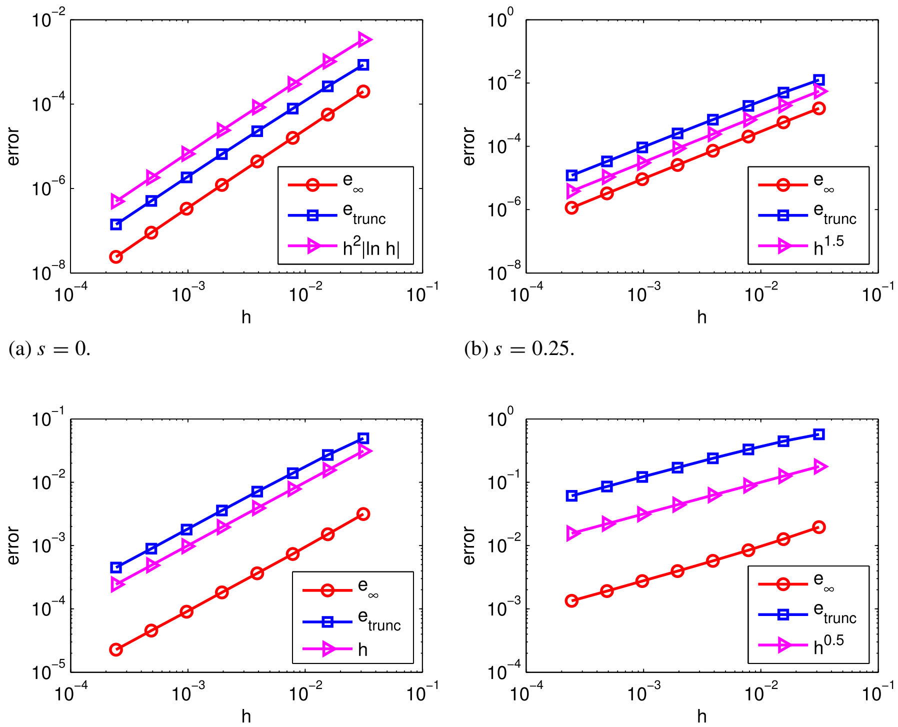
Table 3
Example 2: Numerical results for $\|\mathbb{C}^{-1}\|_\infty$ for the collocation scheme for the pure nonlocal diffusion problem, i.e., $\lambda \equiv 0$ in (31).

h	$s = 0$	$s = 0.25$	$s = 0.5$	$s = 0.75$
$1/2^5$	1.541	0.722	0.375	0.151
$1/2^6$	1.643	0.779	0.363	0.135
$1/2^7$	1.698	0.811	0.357	0.125
$1/2^8$	1.727	0.827	0.353	0.119
$1/2^9$	1.741	0.836	0.351	0.115
$1/2^{10}$	1.749	0.840	0.350	0.112
$1/2^{11}$	1.752	0.842	0.349	0.110
$1/2^{12}$	1.754	0.843	0.349	0.109

that the condition numbers $\mathrm{Cond}_\infty(\mathbb{C})$ are about $O(\ln n)$ if $s = 0$ and $O(n^{2s})$ if $s \in (0, 1)$, which again agree well with the conclusions in Theorem 9, even though $\lambda_{\min} = 0$ for Lemma 8 in this case.

Example 3. We let $s = 0.75$ and test the collocation scheme for solving the pure nonlocal diffusion problem, i.e., $\lambda \equiv 0$ in (31), along with a standard boundary condition (instead of a volume constraint boundary condition) as discussed in Remark 3. We set $a = 0$, $b = 1$, and $\lambda \equiv 0$ and take the same $u(x)$ used in Examples 1 and 2 as the exact solution.

As discussed in Remark 3, we could not prove convergence of the proposed collocation scheme for this problem. However, from the numerical results presented in Table 4 and Fig. 8, we see that $\|\mathbb{C}^{-1}\|_\infty$ still remains bounded which yields the optimal convergence rates of the maximum nodal errors and truncation errors, but the increase in the value of the condition number $\mathrm{Cond}_\infty(\mathbb{C})$ seems to be less than $O(n^{2s})$ if $s > 0.5$.



(a) $s = 0$.

(b) $s = 0.25$.

(c) $s = 0.5$.

(d) $s = 0.75$.

Fig. 6. Example 2: Maximum nodal errors and truncation errors for the collocation scheme for the pure nonlocal diffusion problem, i.e., $\lambda \equiv 0$ in (31).

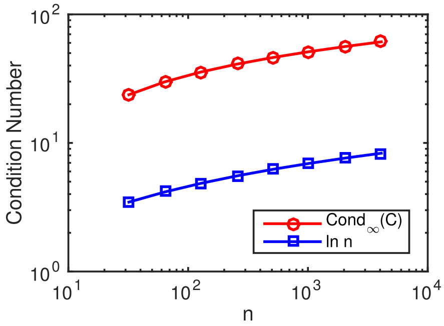
Table 4
Example 3: Numerical results for $\|\mathbb{C}^{-1}\|_\infty$ for the collocation scheme for the pure nonlocal diffusion problem, i.e., $\lambda \equiv 0$ in (31), with standard boundary conditions.

h	$1/2^5$	$1/2^6$	$1/2^7$	$1/2^8$	$1/2^9$	$1/2^{10}$	$1/2^{11}$	$1/2^{12}$
$\|\mathbb{C}^{-1}\|_\infty$	0.416	0.357	0.303	0.264	0.235	0.213	0.198	0.186

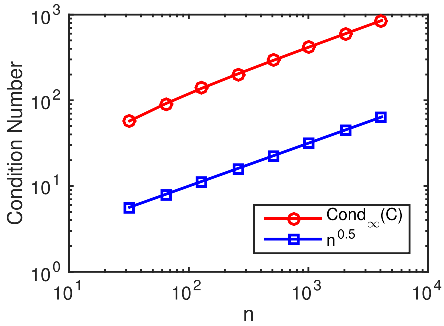
6. Concluding remarks

In recent years, there has been tremendous interest in developing various discretization methods to numerically simulate peridynamic and nonlocal diffusion models, including finite difference methods [16,22], finite element methods [23–26], and collocation methods [18]. Finite element methods, as a classic and popular numerical method, can not only make error analysis easy to accomplish but also produce satisfactory accuracy for the nonlocal diffusion model. On the other hand, in practice it suffers a serious computational obstacle: each entry in the stiffness matrix requires a double integration process which renders the assembly of the stiffness matrix much more expensive. Compared to FEM, the implementation of collocation methods is easier due to the requirement of just a single integration for computing each entry of the stiffness matrix.

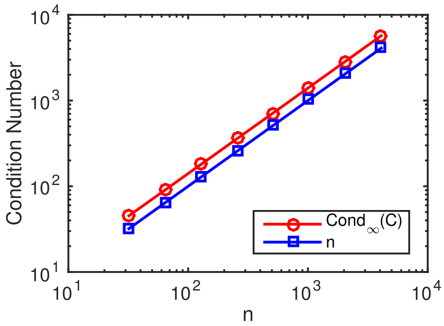
The main contributions of this paper are to develop nodal-type quadrature rules for evaluating finite-part integrals, and subsequently construct collocation methods for solving finite-part integral equations and nonlocal diffusion models in one dimension. The concept of Hadamard finite part integrals was introduced to build up the nodal-type quadrature rules, and then error estimates were given on both uniform and nonuniform grids. When $s \geq 0.5$, we also improve the quadrature rules by some modifications, to not only make the error estimate feasible but also guarantee stability for the collocation schemes. We also rigorously analyze the proposed collocation approximations



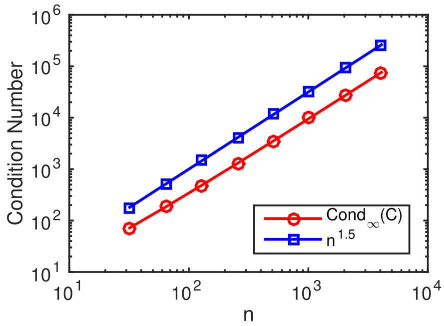
(a) $s = 0$.



(b) $s = 0.25$.



(c) $s = 0.5$.



(d) $s = 0.75$.

Fig. 7. Example 2: Condition numbers of the stiffness matrix of the collocation scheme for the pure nonlocal diffusion problem, i.e., $\lambda \equiv 0$ in (31).

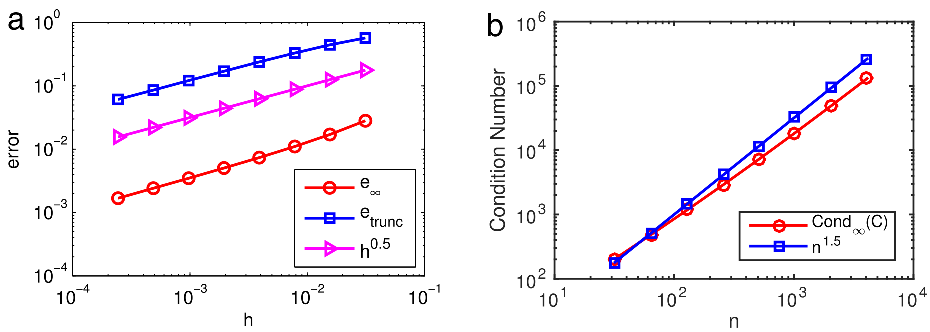


Fig. 8. Example 3: (a) Maximum nodal errors and truncation errors, and (b) condition numbers of the stiffness matrix of the collocation scheme for the pure nonlocal diffusion problem, i.e., $\lambda \equiv 0$ in (31), with the standard boundary condition.

and establish their solvability, uniqueness, and optimal error estimates. The error analysis for the pure nonlocal diffusion model with standard boundary conditions and extension of these solution techniques to problems in higher dimensional spaces remain for future investigations.

To extend the proposed methods to higher dimensions, we need develop efficient and effective nodal-type quadrature rules for evaluating integrals of the type

$$\int_{\Omega} \frac{u(\boldsymbol{y})}{\|\boldsymbol{x}-\boldsymbol{y}\|^{d+2s}}\, d\boldsymbol{y} \tag{48}$$

where $\Omega \subset \mathbb{R}^d$ with $d = 2, 3$, and then apply the corresponding collocation schemes for solving the higher dimensional finite-part integral equations and nonlocal diffusion equations. The main difficulties lie in the dimension, the irregularity of the integration domain and the higher-order singularity of the kernel function, as well as the type of the meshes. The techniques in this paper could be easily extended to construct nodal-type quadrature rules on the rectangular meshes in higher dimensions, which is currently under our investigation. However, if a triangular/tetrahedral mesh is used, more challenges will be encountered due to the complicated integration regions.

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