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An efficient calculation of the Clausen functions $\mathrm{Cl}_n(\theta)(n \geq 2)$

Jiming Wu · Xiaoping Zhang · Dongjie Liu

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Abstract The Clausen functions appear in many problems, such as in the computation of singular integrals, quantum field theory, and so on. In this paper, we consider the Clausen functions $\mathrm{Cl}_n(\theta)$ with $n \geq 2$. An efficient algorithm for evaluating them is suggested and the corresponding convergence analysis is established. Finally, some numerical examples are presented to show the efficiency of our algorithm.

Keywords Clausen functions · Error estimate

Mathematics Subject Classification (2000) 65R99 · 65G20 · 65B10

1 Introduction

In many occasions, such as in the evaluation of singular integrals [1, 4, 6, 8, 14, 15, 17] and quantum field theory [5, 7, 9], it is often necessary to compute the Clausen

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J. Wu
Laboratory of Computational Physics, Institute of Applied Physics and Computational Mathematics, P.O. Box 8009, Beijing 100088, P.R. China
e-mail: wu_jiming@iapcm.ac.cn

X. Zhang (✉)
School of Mathematics and Statistics, Wuhan University, Wuhan 430072, P.R. China
e-mail: xpzhang@lsec.cc.ac.cn

D. Liu
Department of Mathematics, College of Sciences, Shanghai University, Shanghai 200444, P.R. China
e-mail: liudj@shu.edu.cn

functions, defined by

$$\mathrm{Cl}_n(\theta) = \begin{cases} \sum_{k=1}^{\infty} \frac{\sin(k\theta)}{k^n}, & n \text{ even}, \\ \sum_{k=1}^{\infty} \frac{\cos(k\theta)}{k^n}, & n \text{ odd}. \end{cases} \tag{1.1}$$

For $n = 1$, the function has an equivalent form

$$\mathrm{Cl}_1(\theta) = -\ln\left|2\sin\frac{\theta}{2}\right| \tag{1.2}$$

and for $n = 2$, it becomes the Clausen's integral

$$\mathrm{Cl}_2(\theta) = -\int_0^{\theta} \ln\left|2\sin\frac{t}{2}\right| dt. \tag{1.3}$$

Moreover,

$$\mathrm{Cl}_n(0) = \begin{cases} 0, & n \text{ even}, \\ \zeta(n), & n \text{ odd}, \end{cases} \tag{1.4}$$

where

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$$

is the Riemann zeta function.

We list some properties of the Clausen functions in the next theorem, which can be obtained directly from the definition (1.1).

Theorem 1.1 (1) (Recurrence relations) *For any positive integer n*,

$$\mathrm{Cl}'_{n+1}(\theta) = (-1)^{n+1}\mathrm{Cl}_n(\theta). \tag{1.5}$$

(2) (Periodicity) *For any positive integer n and any integer m*,

$$\mathrm{Cl}_n(\theta) = \mathrm{Cl}_n(\theta + 2m\pi). \tag{1.6}$$

(3) (Parity) *For any positive integer n*,

$$\mathrm{Cl}_n(-\theta) = (-1)^{n+1}\mathrm{Cl}_n(\theta). \tag{1.7}$$

Remark 1.1 Because of the periodicity in (1.6) and the parity in (1.7), the computation of the Clausen functions can be restricted to the interval $[0, \pi]$. Therefore, we just need to proceed in our analysis with the condition $\theta \in [0, \pi]$.

By (1.2), we see that $\mathrm{Cl}_1(\theta)$ has an explicitly simple form. Unfortunately, the Clausen functions $\mathrm{Cl}_n(\theta)$ $(n \geq 2)$ do not possess this property. However, since the series in (1.1) with $n \geq 2$ is convergent, it's natural for us to propose the following direct algorithm.

Algorithm 1 Let N be a positive integer. Then for $n \geq 2$, one can use the truncated function

$$\mathrm{Cl}_n^N(\theta) = \begin{cases} \sum_{k=1}^N \frac{\sin(k\theta)}{k^n}, & n \text{ even}, \\ \sum_{k=1}^N \frac{\cos(k\theta)}{k^n}, & n \text{ odd} \end{cases} \tag{1.8}$$

to approximate $\mathrm{Cl}_n(\theta)$.

Algorithm 1 has the following error estimate.

Theorem 1.2 *Assume that $\mathrm{Cl}_n^N(\theta)$ is defined by (1.8). Then there exists a positive constant C, independent of N, such that*

$$|\mathrm{Cl}_n^N(\theta) - \mathrm{Cl}_n(\theta)| \leq \frac{C}{nN^{n-1}}.$$

Proof For $k \geq N+1$, we see that

$$\begin{aligned} \frac{1}{k^{2n_1}} < \frac{1}{k^{n_1}(k-1)^{n_1}} &= \frac{1}{k^{n_1} - (k-1)^{n_1}} \left[\frac{1}{(k-1)^{n_1}} - \frac{1}{k^{n_1}} \right] \\ &\leq \frac{1}{(N+1)^{n_1} - N^{n_1}} \left[\frac{1}{(k-1)^{n_1}} - \frac{1}{k^{n_1}} \right] \\ &\leq \frac{1}{n_1 N^{n_1-1}} \left[\frac{1}{(k-1)^{n_1}} - \frac{1}{k^{n_1}} \right]. \end{aligned} \tag{1.9}$$

Therefore, if $n = 2n_1$, by (1.9),

$$\begin{aligned} |\mathrm{Cl}_n^N(\theta) - \mathrm{Cl}_n(\theta)| &\leq \sum_{k=N+1}^{\infty} \frac{1}{k^{2n_1}} \\ &\leq \frac{1}{n_1 N^{n_1-1}} \sum_{k=N+1}^{\infty} \left[\frac{1}{(k-1)^{n_1}} - \frac{1}{k^{n_1}} \right] \\ &\leq \frac{1}{n_1 N^{2n_1-1}} = \frac{2}{nN^{n-1}}. \end{aligned}$$

The case where $n = 2n_1 + 1$ can be proved similarly, which completes the proof. □

Theorem 1.2 shows that the direct algorithm for $\mathrm{Cl}_2(\theta)$ has the slowest convergence rate, compared with that for the rest of the Clausen functions, and the convergence rate is improved with the increase in n. For this reason, there appeared a lot of papers discussing the computation of the Clausen's integral $\mathrm{Cl}_2(\theta)$. For example, de Doelder [10] suggested an efficient algorithm for $\mathrm{Cl}_2(\theta)$ when θ is equal to a rational multiple of π. The starting point was an integral representation of $\mathrm{Cl}_2(\theta)$, which is different from, but equivalent to (1.1) with $n = 2$, i.e.,

$$\mathrm{Cl}_2(\theta) = -\sin\theta \int_0^1 \frac{\ln \rho}{\rho^2 - 2\rho\cos\theta + 1}\, d\rho, \quad 0 < \theta < 2\pi.$$

Concurrently, Grosjean [11] discussed the same problem through a different approach by making use of the series expansion

$$\mathrm{Cl}_2(p\pi/q) = \sum_{l=1}^{\infty} \frac{\sin(lp\pi/q)}{l^2}$$

and proposed two computational schemes for $\mathrm{Cl}_2(\theta)$. The first one is

$$\begin{aligned}\mathrm{Cl}_2(\theta) \approx{}& -\theta \ln\left(\sin\frac{\theta}{2}\right) + \frac{\theta}{24}(\pi^2 - \theta^2) - \left(\frac{3}{2} - 2\ln 2\right)\sin\theta \\ &+ \left(\ln 2 - \frac{11}{16}\right)\sin(2\theta), \quad 0 \le \theta \le \pi,\end{aligned} \tag{1.10}$$

whose relative error does not exceed 0.63%, and the second one is

$$\begin{aligned}\mathrm{Cl}_2(\theta) \approx{}& -\theta \ln\left(\sin\frac{\theta}{2}\right) + \frac{\theta}{2880}(\pi^2 - \theta^2)(120 - 7\pi^2 + 3\theta^2) + \left(2\ln 2 - \frac{5}{4}\right)\sin\theta \\ &- \left(\frac{89}{128} - \ln 2\right)\sin(2\theta) + \left(\frac{2}{3}\ln 2 - \frac{449}{972}\right)\sin(3\theta) \\ &- \left(\frac{4259}{12288} - \frac{1}{2}\ln 2\right)\sin(4\theta) + \left(\frac{2}{5}\ln 2 - \frac{10397}{37500}\right)\sin(5\theta)\end{aligned} \tag{1.11}$$

with a relative error less than 0.003%, where $0 \le \theta \le \pi$.

There exist some shortcomings in the methods of de Doelder and Grosjean. The main one is that their methods are confined to the case where θ is equal to a rational multiple of π belonging to $[0, 2\pi]$. Moreover, (1.10) and (1.11) are not accurate enough to compute $\mathrm{Cl}_2(\theta)$.

In 1968, Wood [16] suggested an efficient calculation of $\mathrm{Cl}_2(\theta)$ by integrating by parts the integral in (1.3) and then making use of the Chebyshev expansion for $t\cot\frac{\pi t}{2}$, which leads to the identity

$$\mathrm{Cl}_2(\theta) = -\theta \ln\left(2\sin\frac{\theta}{2}\right) + \frac{1}{2}\pi^2 N_0(\theta), \quad 0 \le \theta \le \pi$$

with

$$N_0(\pi x) = x \sum_{r=0}^{\infty}{}' a_{2r} T_{2r}(x), \quad 0 \le x \le 1$$

and

$$a_{2r} = \sum_{n=r}^{\infty} \frac{(-1)^n B_{2n} \pi^{2n-1}}{(2n+1)4^{n-1}(n-r)!(n+r)!},$$

where $\sum'$ indicates the first term is halved, $T_{2r}(x)$ are the Chebyshev polynomials of the first kind and B_{2n} are the Bernoulli numbers. Since the series a_{2r} converges rapidly, an accuracy of 20 digits can be achieved by only requiring the calculation of a sine and a logarithm function and the evaluation of a Chebyshev series with 17 terms. In 1995, Kölbig [13] presented a different, substantially faster algorithm for $\mathrm{Cl}_2(\theta)$, also based on the Chebyshev expansion, which can be represented as the following two formulae:

$$\mathrm{Cl}_2(\theta) = \theta - \theta \ln|\theta| + \frac{1}{2}\theta^3 \sum_{n=0}^{\infty} a_n T_{2n}\left(\frac{2\theta}{\pi}\right), \quad -\frac{1}{2}\pi \leq \theta \leq \frac{1}{2}\pi$$

and

$$\mathrm{Cl}_2(\theta) = (\pi - \theta) \sum_{n=0}^{\infty} b_n T_{2n}\left(\frac{2\theta}{\pi} - 2\right), \quad \frac{1}{2}\pi \leq \theta \leq \frac{3}{2}\pi,$$

where a_n and b_n are some coefficients, whose definitions and evaluation constitute the main contents of [13].

To our knowledge, there apparently are few works in the literature that address the efficient computation of Clausen functions with $n > 2$. Recently, we were informed that a recurrence algorithm for the Clausen functions with $n \geq 2$ can be constructed through Entry 13 in page 260 of [3]. In this paper, we obtain a different series, but equivalent to (1.1), for the Clausen functions $\mathrm{Cl}_n(\theta)$ $(n \geq 2)$, which is not of the recurrence type and has an exponential convergence rate. Thus, an efficient algorithm is naturally proposed. Compared with the aforementioned recurrence algorithm, our formula is closed and can be used to compute the Clausen functions $\mathrm{Cl}_n(\theta)$ with different n directly.

The rest of the paper is organized as follows. In the next section, the main result of this paper is suggested and the related theoretical analysis is given. In Sect. 3, some numerical experiments are presented to show the efficiency of our algorithm.

2 Calculation of $\mathrm{Cl}_n(\theta)$ $(n \geq 2)$

Define

$$\mathscr{L}_n(\theta) := -\int_0^{\theta} t^n \ln\left|2\sin\frac{t}{2}\right| dt \tag{2.1}$$

and

$$\mathscr{J}_n(\theta) := \int_0^{\theta} t^n \cot\frac{t}{2}\, dt. \tag{2.2}$$

By applying the identity

$$\frac{t}{2}\cot\frac{t}{2} = 1 + \sum_{k=1}^{\infty} (-1)^k B_{2k} \frac{t^{2k}}{(2k)!}$$

in (2.2), we get

$$\mathscr{J}_{n+1}(\theta) = 2(n+1)\mathscr{N}_n(\theta), \tag{2.3}$$

where

$$\mathcal{N}_n(\theta) = \frac{1}{n+1}\left[\frac{\theta^{n+1}}{n+1} + \sum_{k=1}^{\infty}(-1)^k B_{2k}\frac{\theta^{2k+n+1}}{(2k+n+1)(2k)!}\right]. \tag{2.4}$$

By integration by parts of (2.1) and using (2.3), we have

$$\begin{aligned}\mathcal{L}_n(\theta) &= -\frac{\theta^{n+1}}{n+1}\ln\left|2\sin\frac{\theta}{2}\right| + \frac{1}{2(n+1)}\mathcal{J}_{n+1}(\theta)\\ &= -\frac{\theta^{n+1}}{n+1}\ln\left|2\sin\frac{\theta}{2}\right| + \mathcal{N}_n(\theta).\end{aligned} \tag{2.5}$$

The following lemma will be employed in proving the main result presented in Theorem 2.1.

Lemma 2.1 *Assume that m, n are two positive integers, and let $\mathcal{N}_n$ be defined in* (2.4). *Then we have*

$$\int_0^\theta t^m \mathcal{N}_n'(t)\,dt = \frac{n+m+1}{n+1}\mathcal{N}_{n+m}(\theta) \tag{2.6}$$

and

$$\int_0^\theta t^m \mathcal{N}_n(t)\,dt = \frac{1}{m+1}\left[\theta^{m+1}\mathcal{N}_n(\theta) - \frac{n+m+2}{n+1}\mathcal{N}_{n+m+1}(\theta)\right]. \tag{2.7}$$

Proof Taking the derivative of $\mathcal{N}_n(t)$ and then multiplying by t^m, we get

$$t^m\mathcal{N}_n'(t) = \frac{1}{n+1}\left[t^{n+m} + \sum_{k=1}^{\infty}(-1)^k B_{2k}\frac{t^{2k+n+m}}{(2k)!}\right].$$

Then,

$$\begin{aligned}\int_0^\theta t^m\mathcal{N}_n'(t)\,dt &= \frac{1}{n+1}\left[\frac{\theta^{n+m+1}}{n+m+1} + \sum_{k=1}^{\infty}(-1)^k B_{2k}\frac{\theta^{2k+n+m+1}}{(2k+n+m+1)(2k)!}\right]\\ &= \frac{n+m+1}{n+1}\mathcal{N}_{n+m}(\theta).\end{aligned}$$

(2.7) can be obtained from (2.6) by only integrating by parts. The proof is completed. □

Theorem 2.1 *For the Clausen functions* $\mathrm{Cl}_n(\theta)(n \geq 2)$ *with* $0 \leq \theta \leq \pi$, *we have*

$$\begin{aligned}\mathrm{Cl}_n(\theta) &= (-1)^{[(n+1)/2]}\frac{\theta^{n-1}}{(n-1)!}\ln\left(2\sin\frac{\theta}{2}\right)\\ &\quad + \frac{(-1)^{[n/2]+1}}{(n-2)!}\sum_{i=0}^{n-2}(-1)^i\binom{n-2}{i}\theta^i\mathcal{N}_{n-2-i}(\theta) + \mathcal{P}_n(\theta),\end{aligned} \tag{2.8}$$

where

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

is the binomial coefficient, $\mathscr{N}_i(\theta)$ *is defined by* (2.4) *and*

$$\mathscr{P}_n(\theta) = \sum_{i=2}^{n} (-1)^{[(n-1)/2]+[(i-1)/2]} \frac{\theta^{n-i}}{(n-i)!} \mathrm{Cl}_i(0)$$

with $\mathrm{Cl}_i(0)$ *defined in* (1.4).

Proof We adopt the approach of mathematical induction.

Initial step. The case where $n = 2$ comes from (2.5) by noting $\mathrm{Cl}_2(\theta) = \mathscr{L}_0(\theta)$.

Inductive Step. Our inductive assumption is: Suppose the result holds for $n = k$, i.e.,

$$\mathrm{Cl}_k(\theta) = (-1)^{[(k+1)/2]} \frac{\theta^{k-1}}{(k-1)!} \ln\left(2\sin\frac{\theta}{2}\right) + \frac{(-1)^{[k/2]+1}}{(k-2)!} \sum_{i=0}^{k-2} (-1)^i \binom{k-2}{i} \theta^i \mathscr{N}_{k-2-i}(\theta) + \mathscr{P}_k(\theta) \tag{2.9}$$

and

$$\mathscr{P}_k(\theta) = \sum_{i=2}^{k} (-1)^{[(k-1)/2]+[(i-1)/2]} \frac{\theta^{k-i}}{(k-i)!} \mathrm{Cl}_i(0). \tag{2.10}$$

We need to prove the formula (2.8) is true for $n = k + 1$. Recalling the recurrence relation (1.5), integrating both sides of (2.9) from 0 to θ and taking into account (2.7), we get

$$\begin{aligned}
\int_0^\theta \mathrm{Cl}'_{k+1}(t)\,dt &= (-1)^{k+1} \int_0^\theta \mathrm{Cl}_k(t)\,dt \\
&= (-1)^{k+1} \left\{ (-1)^{[(k+1)/2]+1} \frac{\mathscr{L}_{k-1}(\theta)}{(k-1)!} \right. \\
&\quad + \frac{(-1)^{[k/2]+1}}{(k-2)!} \sum_{i=0}^{k-2} (-1)^i \binom{k-2}{i} \int_0^\theta t^i \mathscr{N}_{k-2-i}(t)\,dt \\
&\quad \left. + \int_0^\theta \mathscr{P}_k(t)\,dt \right\} \\
&= (-1)^{k+1} \left\{ (-1)^{[(k+1)/2]} \frac{\theta^k}{k!} \ln\left(2\sin\frac{\theta}{2}\right) \right. \\
&\quad + (-1)^{[(k+1)/2]+1} \frac{\mathscr{N}_{k-1}(\theta)}{(k-1)!}
\end{aligned}$$

$$+\frac{(-1)^{[k/2]+1}}{(k-2)!}\sum_{i=0}^{k-2}\frac{(-1)^i}{i+1}\binom{k-2}{i}\Bigg[\theta^{i+1}\mathscr{N}_{k-2-i}(\theta)$$
$$-\frac{k}{k-1-i}\mathscr{N}_{k-1}(\theta)\Bigg]+\int_0^\theta \mathscr{P}_k(t)\,dt\Bigg\}$$

By employing the identities

$$(-1)^{k+1+[(k+1)/2]}=(-1)^{[(k+2)/2]},\qquad (-1)^{k+1+[k/2]}=(-1)^{[(k+1)/2]+1}$$

and

$$\begin{aligned}
&(-1)^{[k/2]+1}\sum_{j=1}^{k-1}(-1)^j\frac{k}{k-j}\binom{k-1}{j}+(-1)^{[(k+1)/2]+1}\\
&=(-1)^{[k/2]+1}\left[\sum_{j=0}^{k}(-1)^j\binom{k}{j}-1-(-1)^k\right]+(-1)^{[(k+1)/2]+1}\\
&=(-1)^{[k/2]+2}+(-1)^{k+[k/2]}+(-1)^{[(k+1)/2]+1}\\
&=(-1)^{[k/2]},
\end{aligned}$$

we have

$$\begin{aligned}
&\mathrm{Cl}_{k+1}(\theta)-\mathrm{Cl}_{k+1}(0)\\
&=(-1)^{k+1}\Bigg\{\frac{(-1)^{[k/2]}}{(k-1)!}\sum_{j=1}^{k-1}(-1)^j\binom{k-1}{j}\theta^j\mathscr{N}_{k-1-j}(\theta)\\
&\quad+\left[(-1)^{[k/2]+1}\sum_{j=1}^{k-1}(-1)^j\frac{k}{k-j}\binom{k-1}{j}+(-1)^{[(k+1)/2]+1}\right]\frac{\mathscr{N}_{k-1}(\theta)}{(k-1)!}\\
&\quad+(-1)^{[(k+1)/2]}\frac{\theta^k}{k!}\ln\left(2\sin\frac{\theta}{2}\right)+\int_0^\theta \mathscr{P}_k(t)\,dt\Bigg\}\\
&=(-1)^{k+1}\Bigg\{\frac{(-1)^{[k/2]}}{(k-1)!}\sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}\theta^j\mathscr{N}_{k-1-j}(\theta)\\
&\quad+(-1)^{[(k+1)/2]}\frac{\theta^k}{k!}\ln\left(2\sin\frac{\theta}{2}\right)+\int_0^\theta \mathscr{P}_k(t)\,dt\Bigg\}\\
&=(-1)^{[(k+1)/2]+1}\frac{1}{(k-1)!}\sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}\theta^j\mathscr{N}_{k-1-j}(\theta)\\
&\quad+(-1)^{[(k+2)/2]}\frac{\theta^k}{k!}\ln\left(2\sin\frac{\theta}{2}\right)+(-1)^{k+1}\int_0^\theta \mathscr{P}_k(t)\,dt.
\end{aligned}$$

It follows that

$$\mathrm{Cl}_{k+1}(\theta) = (-1)^{[(k+1)/2]+1} \frac{1}{(k-1)!} \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} \theta^j \mathscr{N}_{k-1-j}(\theta) + (-1)^{[(k+2)/2]} \frac{\theta^k}{k!} \ln\left(2 \sin\frac{\theta}{2}\right) + (-1)^{k+1} \int_0^\theta \mathscr{P}_k(t)\, dt + \mathrm{Cl}_{k+1}(0).$$

From the definition (2.10) of $\mathscr{P}_k(\theta)$, we can easily see that

$$\begin{aligned}
&(-1)^{k+1} \int_0^\theta \mathscr{P}_k(t)\, dt + \mathrm{Cl}_{k+1}(0) \\
&= \sum_{i=2}^{k} (-1)^{k+1+[(k-1)/2]+[(i-1)/2]} \frac{\theta^{k-i+1}}{(k-i+1)!} \mathrm{Cl}_i(0) + \mathrm{Cl}_{k+1}(0) \\
&= \sum_{i=2}^{k+1} (-1)^{[k/2]+[(i-1)/2]} \frac{\theta^{k-i+1}}{(k-i+1)!} \mathrm{Cl}_i(0) \\
&= \mathscr{P}_{k+1}(\theta),
\end{aligned}$$

which completes the inductive step. □

Define

$$\mathscr{N}_n^N(\theta) = \frac{1}{n+1} \left[\frac{\theta^{n+1}}{n+1} + \sum_{k=1}^{N} (-1)^k B_{2k} \frac{\theta^{2k+n+1}}{(2k+n+1)(2k)!} \right]. \tag{2.11}$$

Theorem 2.2 *Assume that $\mathscr{N}_n(\theta)$ and $\mathscr{N}_n^N(\theta)$ are defined in (2.4) and (2.11), respectively, where N is a positive integer. Then, for $\theta \in [0, \pi]$, we have*

$$|\mathscr{N}_n^N(\theta) - \mathscr{N}_n(\theta)| \leq \frac{(2\pi)^{n+3}}{6} \frac{1}{2N+n+3} \left(\frac{\theta}{2\pi}\right)^{2N+n+3}.$$

Proof By the definition (2.4) of $\mathscr{N}_n(\theta)$ and the equality (p. 105, [2] or p. 135, [3])

$$B_{2k} = \frac{2(-1)^{k+1} \zeta(2k)(2k)!}{(2\pi)^{2k}}, \tag{2.12}$$

we have

$$\begin{aligned}
|\mathscr{N}_n^N(\theta) - \mathscr{N}_n(\theta)| &= \left| \sum_{k=N+1}^{\infty} (-1)^k B_{2k} \frac{\theta^{2k+n+1}}{(2k+n+1)(2k)!} \right| \\
&= 2(2\pi)^{n+1} \sum_{k=N+1}^{\infty} \frac{\zeta(2k)}{2k+n+1} \left(\frac{\theta}{2\pi}\right)^{2k+n+1}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{(2\pi)^{n+3}}{12} \sum_{k=N+1}^{\infty} \frac{1}{2k+n+1} \left(\frac{\theta}{2\pi}\right)^{2k+n+1} \\
&\leq \frac{(2\pi)^{n+3}}{12} \frac{1}{2N+n+3} \sum_{k=N+1}^{\infty} \left(\frac{\theta}{2\pi}\right)^{2k+n+1} \\
&= \frac{(2\pi)^{n+3}}{12} \frac{1}{1-\frac{\theta}{2\pi}} \frac{1}{2N+n+3} \left(\frac{\theta}{2\pi}\right)^{2N+n+3} \\
&\leq \frac{(2\pi)^{n+3}}{6} \frac{1}{2N+n+3} \left(\frac{\theta}{2\pi}\right)^{2N+n+3},
\end{aligned}
$$

where

$$
\frac{\zeta(2k)}{2k+n+1} \leq \frac{\zeta(2)}{2k+n+1} = \frac{\pi^2}{6(2k+n+1)}
$$

has been used. □

Now, based on Theorem 2.1, we propose a new algorithm to compute the Clausen function $\mathrm{Cl}_n(\theta)(n \geq 2)$.

Algorithm 2 Let N be a positive integer, then for $n \geq 2$, one can use the truncated function

$$
\begin{aligned}
\tilde{\mathrm{Cl}}_n^N(\theta) &= (-1)^{[\frac{n+1}{2}]} \frac{\theta^{n-1}}{(n-1)!} \ln\left(2\sin\frac{\theta}{2}\right) \\
&\quad + \frac{(-1)^{[\frac{n}{2}]+1}}{(n-2)!} \sum_{i=0}^{n-2} (-1)^i \binom{n-2}{i} \theta^i \mathscr{N}_{n-2-i}^N(\theta) + \mathscr{P}_n(\theta) \qquad (2.13)
\end{aligned}
$$

to approximate $\mathrm{Cl}_n(\theta)$ in $[0, \pi]$.

The error estimate of Algorithm 2 is given in the following theorem.

Theorem 2.3 *Let* $\tilde{\mathrm{Cl}}_n^N(\theta)$ *be defined in* (2.13), *we have for* $\theta \in [0, \pi]$ *that*

$$
|\tilde{\mathrm{Cl}}_n^N(\theta) - \mathrm{Cl}_n(\theta)| \leq \frac{(2\pi)^{n+1}}{6(n-2)!} \frac{1}{2N+3} \left(\frac{\theta}{2\pi}\right)^{2N+3}. \qquad (2.14)
$$

Proof From Theorem 2.1 and Theorem 2.2, we get

$$
\begin{aligned}
|\tilde{\mathrm{Cl}}_n^N(\theta) - \mathrm{Cl}_n(\theta)| &\leq \frac{1}{(n-2)!} \sum_{i=0}^{n-2} \binom{n-2}{i} \theta^i |\mathscr{N}_{n-2-i}^N(\theta) - \mathscr{N}_{n-2-i}(\theta)| \\
&\leq \frac{1}{(n-2)!} \sum_{i=0}^{n-2} \binom{n-2}{i} \theta^i \frac{(2\pi)^{n-i+1}}{6(2N+n-i+1)} \left(\frac{\theta}{2\pi}\right)^{2N+n-i+1}
\end{aligned}
$$

$$\leq \frac{(2\pi)^3}{6(n-2)!}\frac{1}{2N+3}\left(\frac{\theta}{2\pi}\right)^{2N+3}\sum_{i=0}^{n-2}\binom{n-2}{i}\theta^i\theta^{n-2-i}$$

$$= \frac{(2\pi)^3}{6(n-2)!}\frac{1}{2N+3}\left(\frac{\theta}{2\pi}\right)^{2N+3}(2\theta)^{n-2}$$

$$\leq \frac{(2\pi)^{n+1}}{6(n-2)!}\frac{1}{2N+3}\left(\frac{\theta}{2\pi}\right)^{2N+3},$$

where the binomial theorem has been used. □

Remark 2.1 From (2.8), we can rewrite the Clausen's integral $\mathrm{Cl}_2(\theta)$ as the form of

$$\mathrm{Cl}_2(\theta) = -\theta\ln\left(\sin\frac{\theta}{2}\right) + (1-\ln 2)\theta + \sum_{k=1}^{\infty}(-1)^k B_{2k}\frac{\theta^{2k+1}}{(2k+1)!}. \tag{2.15}$$

Especially,

$$\tilde{\mathrm{Cl}}_2^2(\theta) = -\theta\ln\left(\sin\frac{\theta}{2}\right) + (1-\ln 2)\theta - \frac{\theta^3}{36} - \frac{\theta^5}{3600}, \tag{2.16}$$

by setting $N = 2$ and $\theta = \pi/2$ in (2.14), we see that its relative error is less than 0.03605%, which is better than that of (1.10) in $[0, \pi/2]$. If we choose $N = 3$ and $\theta = \pi/2$, we get

$$\tilde{\mathrm{Cl}}_2^3(\theta) = -\theta\ln\left(\sin\frac{\theta}{2}\right) + (1-\ln 2)\theta - \frac{\theta^3}{36} - \frac{\theta^5}{3600} - \frac{\theta^7}{211680}, \tag{2.17}$$

whose relative error does not exceed 0.00175%, which is smaller than that of (1.11) in $[0, \pi/2]$. In other words, one may compute $\mathrm{Cl}_2(\theta)$ by (2.16) and (2.17) to get better accuracy in $[0, \pi/2]$ compared with (1.10) and (1.11), respectively.

Remark 2.2 Obviously, our formula (2.8) is for all $n \geq 2$, but it should be noted that the case $n = 2$, i.e., the formula (2.15), is equivalent with the formula (54.5.4) in page 356 of [12], which has the form

$$\sum_{k=1}^{\infty}\frac{1}{2k+1}t^{2k}\zeta(2k) = \frac{1}{2} - \frac{1}{2}\ln(2\sin\pi t) - \frac{1}{4\pi t}\mathrm{Cl}_2(2\pi t). \tag{2.18}$$

For the comparison with (2.15), we rewrite the above formula as

$$\mathrm{Cl}_2(2\pi t) = 2\pi t - 2\pi t\ln(2\sin\pi t) - 4\pi\sum_{k=1}^{\infty}\frac{1}{2k+1}t^{2k+1}\zeta(2k).$$

Let $\theta = 2\pi t$, we get

$$\mathrm{Cl}_2(\theta) = \theta - \theta\ln\left(2\sin\frac{\theta}{2}\right) - \sum_{k=1}^{\infty}\frac{2\theta^{2k+1}}{(2k+1)(2\pi)^{2k}}\zeta(2k). \tag{2.19}$$

Table 1 Computational results for $\mathrm{Cl}_2(\pi/3)$

N	Wood's method	our method
2	1.01503 35715 95451 19237	1.01494 82728 50627 10187
4	1.01494 22904 33421 03928	1.01494 16096 37502 47813
6	1.01494 16063 89935 32083	1.01494 16064 11480 28148
8	1.01494 16063 99178 12381	1.01494 16064 09654 73839
10	1.01494 16064 09635 20675	1.01494 16064 09653 62573
12	1.01494 16064 09653 78628	1.01494 16064 09653 62502
14	1.01494 16064 09653 62568	1.01494 16064 09653 62502
16	1.01494 16064 09653 62501	1.01494 16064 09653 62502

Table 2 Computational results for $\mathrm{Cl}_2(\pi/2)$

N	Wood's method	our method
2	0.91629 71337 41892 53927	0.91608 27004 26416 14282
4	0.91596 51079 04518 60590	0.91596 58820 50368 02397
6	0.91596 55919 85927 46426	0.91596 55950 03067 86059
8	0.91596 55941 93511 97790	0.91596 55941 79769 43790
10	0.91596 55941 77187 70161	0.91596 55941 77227 25323
12	0.91596 55941 77218 84843	0.91596 55941 77219 04248
14	0.91596 55941 77219 01641	0.91596 55941 77219 01514
16	0.91596 55941 77219 01505	0.91596 55941 77219 01505

While (2.15) can be rewritten as

$$\mathrm{Cl}_2(\theta) = \theta - \theta \ln\left(2\sin\frac{\theta}{2}\right) + \sum_{k=1}^{\infty}(-1)^k B_{2k}\frac{\theta^{2k+1}}{(2k+1)!}. \tag{2.20}$$

Substituting the equality (2.12) into (2.20), we arrive at (2.19), an equivalent counterpart of (2.18).

3 Numerical experiments

In this section, computational results are reported by some examples to confirm our theoretical analysis and to show the efficiency of the algorithm.

Example 3.1 The maximum of $\mathrm{Cl}_2(\theta)$ in $(0,\pi)$ occurs at $\theta = \pi/3$ and has the value

$$\mathrm{Cl}_2\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{6}\left[\psi'\left(\frac{1}{3}\right) - \frac{2}{3}\pi^2\right] = 1.01494\ 16064\ 09653\ 62502\ \ldots,$$

where

$$\psi(z) = \frac{d\log\Gamma(z)}{dz}$$

Table 3 Comparison of CPU time between Wood's method and our method (Unit: second)

$N = 30$		$N = 20$	
Wood's method	our method	Wood's method	our method
0.326550	0.004328	0.307290	0.002732

Table 4 Computational results for $\mathrm{Cl}_3(\pi/3)$ and $\mathrm{Cl}_4(\pi/3)$

N	$\mathrm{Cl}_3(\pi/3)$	$\mathrm{Cl}_4(\pi/3)$
1	0.40046 80185 33921 23289	0.91576 78443 96267 06462
2	0.40068 17094 20569 20104	0.91584 54811 66564 68286
3	0.40068 55524 31712 30139	0.91584 68557 53084 17629
4	0.40068 56325 56073 78935	0.91584 68841 84054 76138
5	0.40068 56343 43984 70986	0.91584 68848 15539 19008
6	0.40068 56343 85515 38233	0.91584 68848 30164 99127
7	0.40068 56343 86506 67983	0.91584 68848 30513 41403
8	0.40068 56343 86530 81643	0.91584 68848 30521 88597
9	0.40068 56343 86531 41315	0.91584 68848 30522 09521
10	0.40068 56343 86531 42808	0.91584 68848 30522 10044
11	0.40068 56343 86531 42846	0.91584 68848 30522 10058
12	0.40068 56343 86531 42847	0.91584 68848 30522 10058
13	0.40068 56343 86531 42847	0.91584 68848 30522 10058

Table 5 Comparison of CPU time between Algorithm 1 and Algorithm 2 with 10 digits accuracy (Unit: second)

$\mathrm{Cl}_3(\pi/3)$		$\mathrm{Cl}_4(\pi/3)$	
Alg 1 (terms: 3011)	Alg 2 (terms: 6)	Alg 1 (terms: 322)	Alg 2 (terms: 5)
0.117010	0.001528	0.012910	0.001999

is the di-gamma function, $\Gamma(z)$ is the gamma function and

$$\psi'(z) = \sum_{m=0}^{\infty} \frac{1}{(m+z)^2}.$$

Moreover, we see that $\mathrm{Cl}_2(\pi/2) = G$, where $G = 0.91596\,55941\,77219\,01505\,\ldots$ is the Catalan constant. Here, we use Algorithm 2 to compute $\mathrm{Cl}_2(\pi/3)$ and $\mathrm{Cl}_2(\pi/2)$, and the results are presented in Table 1 and Table 2, respectively. From Table 1 and Table 2, we can see that the accuracy of 20 digits can be achieved if we truncate the function $\mathscr{N}_n$ as 12 terms when $\theta = \pi/3$ and 16 terms when $\theta = \pi/2$, which shows that the present algorithm is competitive with Wood's method [16] and also agrees quite well with the estimate in Theorem 2.1. Table 3 describes the comparison of CPU time between Wood's method and our method.

Example 3.2 Here we consider $\mathrm{Cl}_3(\theta)$ and $\mathrm{Cl}_4(\theta)$ to show further the efficiency of our method. Table 4 presents the values of $\mathrm{Cl}_3(\pi/3)$ and $\mathrm{Cl}_4(\pi/3)$ with different truncation terms, which shows that the accuracy of 20 digits can be achieved if about 11 terms are retained. Table 5 describe the comparison of CPU time between Algorithm 1 and Algorithm 2, and the term in the parentheses is the smallest one such that an accuracy of 10 digits is achieved.

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