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L_2 error estimates of collocation methods for solving certain singular integral equations

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ABSTRACT

In this paper, we propose some special collocation schemes for solving Hilbert type singular and hypersingular integral equations on a circle, based on the superconvergence of quadrature rules for evaluating the corresponding singular integrals. With the aid of spectral analysis, the optimal and suboptimal L_2 error estimates are established in a unified framework. At last, several numerical examples are provided to confirm the theoretical analysis.

1. Introduction

Consider the singular integral equation

$$\mathcal{H}\varphi(s) = \int_0^{2\pi} \kappa(t-s)\varphi(t)\,dt = g(s), \quad s \in (0, 2\pi), \tag{1}$$

where $g(s)$ is a given function, $\varphi(t)$ is unknown density function, $\kappa(t)$ is a singular integral kernel, may be $\frac{1}{2\pi}\cot\frac{t}{2}$ or $\frac{1}{4\pi\sin^2(t/2)}$ and all of them are 2π-periodic. If $\kappa(t) = \frac{1}{2\pi}\cot\frac{t}{2}$, (1) is often called as Hilbert singular integral equation, and the integral in the left side must be understood in Cauchy principal value sense. If $\kappa(t) = \frac{1}{4\pi\sin^2(t/2)}$, (1) becomes a hypersingular integral equation, and the integral in the left side must be understood in Hadamard finite part sense. Such equations are frequently encountered in physical and engineering applications, such as in fracture mechanics, elasticity problems, aerodynamics as well as electromagnetic scattering [6,7].

Numerous works have been devoted in developing efficient numerical methods for the solution of such equations. For example, Chandler [3] studied a Cauchy singular integral equation on a smooth closed curve by using a simple midpoint collocation method, and some convergence results, but not optimal, were obtained. By using spectral analysis, Yan [12,13] provided the convergence analysis of the midpoint collocation method for solving boundary integral equations with logarithmic kernel on a closed boundary. Based on trigonometric interpolation, a fully discrete method was suggested for solving the hypersingular boundary integral equation arising from scattering problem by Kress [6], where an exponential convergence rate was proved for analytic boundaries and boundary data. Mülthei and Schenider [10] proposed a fully discrete collocation scheme on graded mesh for solving Hilbert singular integral equations, and some convergence results in sup-norms and weighted L_2-norms were obtained. Schneider investigated the stability of the pseudo-inverse of discretized Hilbert transform in [11]. In [4], a unified framework for various collocation methods of numerical solutions of Hilbert singular integral equations were established. A fast Fourier–Galerkin method for solving a class of singular boundary integral equations was developed in [2] by compressing the dense Galerkin matrix to a sparse one. In [5], the authors studied a collocation scheme based

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on the superconvergence result of midpoint rule for solving hypersingular integral equation on a circle, where the optimal error estimate was established by analyzing the properties of the scheme's coefficient matrix.

In this paper, we extend the spectral analysis proposed in [12] to analyze the collocation method based on the superconvergence results of midpoint and trapezoidal rules for solving (1). Superconvergence analysis of Newton–Cotes rules for singular integrals has been investigated in depth for the past few years [9,15,16,19]. However, up to now, most of these are restricted on the analysis of the quadrature rules for singular integrals. Only a few error estimates have been obtained for the solution of singluar integral equations, such as in [5,17], where the optimal convergence rates in maximum norm were obtained, but the analytic process rely on the properties of the resulted matrix too much. Through the spectral analysis, we investigate the error estimates in a unified framework, where the eigenvalues of the considered schemes can be expressed explicitly, and thus the spectral norm of the resulted coefficient matrices can be bounded. Combined with the superconvergence results of the quadrature rules, the optimal and suboptimal discrete L_2 error estimates are established.

The rest of this paper is organized as follows. In Section 2, we propose the collocation schemes for the solution of (1) in a unified way and then the general result is given. The error estimates are given for Hilbert singular integral equation and hypersingular integral equation in Sections 3 and 4, respectively. Then, we provide the proof of superconvergence results of midpoint rule for evaluating Hilbert singular integrals in Section 5. Finally, several numerical examples are presented to confirm our theoretical analysis.

2. General result

For simplicity of exposition, we confine ourselves to the case where $0 = t_0 < t_1 < t_2 < \cdots < t_n = 2\pi$ is a uniform mesh of $[0, 2\pi]$ with the mesh size $h = 2\pi/n$. Denote the interpolation of φ by

$$\Pi_h\varphi(t) = \sum_{j=0}^{n-1}\varphi(p_j)\chi_j(t), \tag{2}$$

where $p_j \in [t_j, t_{j+1})$ denote the interpolation points and χ_j the corresponding basis function. For example, if p_j is chosen as $\hat{t}_j = \frac{t_j+t_{j+1}}{2}$ and

$$\chi_j(t) = \begin{cases} 1, & \text{on } [t_j, t_{j+1}], \\ 0, & \text{otherwise}, \end{cases} \tag{3}$$

$\Pi_h\varphi$ is the piecewise constant interpolation of φ; if p_j is chosen as t_j and

$$\chi_j(t) = \begin{cases} \frac{t-t_{j-1}}{h}, & \text{on } [t_{j-1}, t_j], \\ \frac{t_{j+1}-t}{h}, & \text{on } [t_j, t_{j+1}], \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

with $t_{-1} + 2\pi = t_{n-1}$, then $\Pi_h\varphi$ is the piecewise linear interpolation of φ. Substituting $\Pi_h\varphi$ with φ defined in (1) yields the quadrature rule of $\mathcal{H}\varphi$

$$\mathcal{Q}_h\varphi(s) = \int_0^{2\pi}\kappa(t-s)\Pi_h\varphi(t)\,dt = \sum_{j=0}^{n-1}\varpi_j(s)\varphi(p_j) = \mathcal{H}\varphi(s) + \mathcal{E}\varphi(s), \tag{5}$$

where $\mathcal{E}\varphi$ denotes the error functional and

$$\varpi_j(s) = \int_0^{2\pi}\kappa(t-s)\chi_j(t)\,dt$$

is the quadrature coefficient. Applying the rule (5) to approximate the integral in (1), we get

$$\sum_{j=0}^{n-1}\varpi_j(s)\varphi(p_j) \approx g(s). \tag{6}$$

Then, collocating this equation at the points $s_i \in (t_i, t_{i+1})$ yields

$$\sum_{j=0}^{n-1}\varpi_j(s_i)\varphi_j = g(s_i), \quad i = 0, 1, \ldots, n-1, \tag{7}$$

where φ_j denote the approximate value of φ at the point p_j. For the sake of analysis, we rewrite it into the matrix form

$$\mathbb{A}\boldsymbol{\varphi}^a = \mathbf{g}^e, \tag{8}$$

where

$$\begin{aligned}&\mathbb{A} = (a_{i,j})_{n\times n}, \quad a_{i,j} = \varpi_j(s_i),\\ &\boldsymbol{\varphi}^a = (\varphi_0, \varphi_1, \ldots, \varphi_{n-1})^T, \quad \mathbf{g}^e = (g(s_0), g(s_1), \ldots, g(s_{n-1}))^T.\end{aligned} \tag{9}$$

Since the kernel κ is $2\pi-$periodic, it is easy to show that the above system is singular. In order to get a well-conditioned system, we adopt the following regularization process by introducing a factor γ_n:

$$\begin{cases} \gamma_n + \sum_{j=0}^{n-1} \varpi_j(s_i)\varphi_j = g(s_i), \quad i = 0, 1, \ldots, n-1, \\ \sum_{j=0}^{n-1} \varphi_j = 0, \end{cases} \tag{10}$$

which can be represented as

$$\bar{\mathbb{A}}\bar{\boldsymbol{\varphi}}^a = \bar{\mathbf{g}}^e, \tag{11}$$

where

$$\bar{\mathbb{A}} = \begin{pmatrix} 0 & \mathbf{e}_n^T \\ \mathbf{e}_n & \mathbb{A} \end{pmatrix}, \quad \bar{\boldsymbol{\varphi}}^a = \begin{pmatrix} \gamma_n \\ \boldsymbol{\varphi}^a \end{pmatrix}, \quad \bar{\mathbf{g}}^e = \begin{pmatrix} 0 \\ \mathbf{g}^e \end{pmatrix} \tag{12}$$

and $\mathbf{e}_n = \Big(\underbrace{1, 1, \ldots, 1}_{n}\Big)_T$.

Theorem 1. *Assume that $\bar{\mathbb{A}}$ is invertible, for the above collocation scheme (10) or (11), the following estimate holds:*

$$\left|\sum_{i=0}^{n-1} |\varphi(p_i) - \varphi_i|^2 h\right| \leqslant \|\bar{\mathbb{A}}^{-1}\|_2 \left[h^{-1}\left(\sum_{j=0}^{n-1}\varphi(p_j)h\right)^2 + \sum_{i=0}^{n-1}|\mathcal{E}\varphi(s_i)|^2 h\right], \tag{13}$$

where $\mathcal{E}\varphi$ is the error functional of the quadrature rule $\mathcal{Q}_h\varphi$ defined by (5).

Proof. Let $\bar{\boldsymbol{\varphi}}^e = (0, \varphi(p_0), \varphi(p_1), \ldots, \varphi(p_{n-1}))^T$ be the exact vector, then we have

$$\bar{\boldsymbol{\varphi}}^e - \bar{\boldsymbol{\varphi}}^a = \bar{\mathbb{A}}^{-1}(\bar{\mathbb{A}}\bar{\boldsymbol{\varphi}}^e - \bar{\mathbf{g}}^e),$$

which implies

$$\|\bar{\boldsymbol{\varphi}}^e - \bar{\boldsymbol{\varphi}}^a\|_2 \leqslant \|\bar{\mathbb{A}}^{-1}\|_2 \|\bar{\mathbb{A}}\bar{\boldsymbol{\varphi}}^e - \bar{\mathbf{g}}^e\|_2. \tag{14}$$

Through direct calculation, we get

$$\bar{\mathbb{A}}\bar{\boldsymbol{\varphi}}^e - \bar{\mathbf{g}}^e = \left(\sum_{j=0}^{n-1}\varphi(p_j), \mathcal{E}\varphi(s_0), \mathcal{E}\varphi(s_1), \ldots, \mathcal{E}\varphi(s_{n-1})\right)^T. \tag{15}$$

The estimate (13) can be obtained immediately from (14) and (15). □

Remark 1. From Theorem 1, it's obvious that to obtain the discrete L_2 estimate of collocation scheme (10), one only need to estimate $\|\bar{\mathbb{A}}^{-1}\|_2$ and $\mathcal{E}\varphi(s)$.

3. Case of $\kappa(t) = \frac{1}{2\pi}\cot\frac{t}{2}$

In this section, we consider the singular integral equation with Hilbert kernel

$$\mathcal{H}_1\varphi(s) := \frac{1}{2\pi}\int_0^{2\pi} \cot\frac{t-s}{2}\varphi(t)\,dt = g(s), \quad s \in (0, 2\pi). \tag{16}$$

Such equation is solvable for $g \in C^\alpha[0, 2\pi](0 < \alpha < 1)$ with $\int_0^{2\pi} g(t)\,dt = 0$. For the uniqueness of a naturally $2\pi-$periodic solution φ, a side condition is required. Here we adopt a periodical condition $\int_0^{2\pi}\varphi(t)\,dt = 0$.

3.1. Midpoint rule and the related collocation scheme

Substituting φ in (1) with its piecewise constant interpolation $\Pi_h\varphi$ defined by (2) with (3) leads to the composite midpoint rule

$$\mathcal{Q}_h\varphi(s) = \frac{1}{2\pi}\int_0^{2\pi}\cot\frac{t-s}{2}\Pi_h\varphi(t)\,dt = \sum_{j=0}^{n-1}\varpi_j(s)\varphi(\hat{t}_j) \tag{17}$$

with

$$\varpi_j(s) = \frac{1}{\pi}\ln\left|\frac{\sin\frac{t_{j+1}-s}{2}}{\sin\frac{t_j-s}{2}}\right|. \tag{18}$$

Before proceeding further, let us introduce some notations for our approach. Throughout this paper, C will denote a generic constant that is independent of h and s and it may have different values in different places. In addition, we assume singular point $s \in (t_m, t_{m+1})$ with some fixed m, i.e., $s = t_m + (1+\tau)h/2$, where $\tau \in (-1,1)$ is local coordinate of the singular point s.

Theorem 2. *Assume that $\mathcal{Q}_h\varphi$ be computed by (17) and (18) with a uniform mesh. Then, for $s = t_m + (1+\tau)h/2\,(0 \leqslant m \leqslant n-1)$ and $\varphi \in C^{2+\alpha}[0,2\pi](0<\alpha<1)$, we have*

$$\mathcal{H}_1\varphi(s) - \mathcal{Q}_h\varphi(s) = \frac{h}{\pi}\varphi'(s)\ln\left(2\cos\frac{\tau\pi}{2}\right) + \mathcal{R}(s), \tag{19}$$

with

$$|\mathcal{R}(s)| \leqslant C\rho(s)\gamma^{-2}(h,s)h^2, \tag{20}$$

where

$$\gamma(h,s) = \min_{0\leqslant i\leqslant n}\frac{|s-t_i|}{h} = \frac{1-|\tau|}{2}, \tag{21}$$

$$\rho(s) = \max_{0\leqslant t\leqslant 2\pi}\kappa_s(t) \tag{22}$$

and

$$\kappa_s(t) = \begin{cases} \frac{(t-s)^2}{\sin^2\frac{t-s}{2}}, & t \neq s, \\ 4, & t = s. \end{cases}$$

The proof of this statement is postponed until Section 5. This theorem gives the error result of composite midpoint rule for evaluating $\mathcal{H}_1\varphi$ defined in (16), which show that for sufficiently smooth function φ, the accuracy of composite midpoint rule is only $O(h)$ in general. However, from (19) we can see that the $O(h)$ term of error expansion vanishes if τ is zeros of the function $\ln\left(2\cos\frac{\tau\pi}{2}\right)$, and thus the superconvergence phenomenon occurs.

Theorem 3. *Assume that $\mathcal{Q}_h\varphi$ be computed by (17) and (18) with a uniform mesh. Then, for $\varphi \in C^{2+\alpha}[0,2\pi](0<\alpha<1)$ at $s^\star = t_m + (1+\tau^\star)h/2(0 \leqslant m \leqslant n-1)$ with $\tau^\star = \pm 2/3$, we have*

$$|\mathcal{E}\varphi(s^\star)| = |\mathcal{H}_1\varphi(s^\star) - \mathcal{Q}_h\varphi(s^\star)| \leqslant Ch^2.$$

Remark 2. For the Newton–Cotes rule for evaluating singular integrals, when the singular point s conincide with some a priori known point, the accuracy may be one order higher than that in general. We refer to this as "pointwise superconvergence phenomenon" for singular integrals. The point where such phenomenon occurs is called as "superconvergence points".

As statements in Theorem 1, in order to achieve better accuracy one should choose the superconvergence points as the collocation points in the collocation scheme. In practice, any one of two series of superconvergence points $\{s_i = t_i + (1+\tau)h/2\}$ with $\tau = 2/3$ or $-2/3$ is recommendable if the middle rule (17) is used to construct the scheme (11).

Next, we study the eigenvalues of systems (8) and (11). It can be easily verified that $\mathbb{A}$ defined by (9) is a circulant matrice. Denote the vector $\mathbf{c} = (c_0, c_1, \ldots, c_{n-1})$ as the first row of $\mathbb{A}$. It's well known that every circulant matrix has eigenvectors

$$\mathbf{v}_j = n^{-1/2}(1, \omega_j, \omega_j^2, \ldots, \omega_j^{n-1})^T, \quad j = 0, 1, \ldots, n-1, \tag{23}$$

where $\omega_j = e^{-2\pi\imath j/n}$ are the n-th roots of unity and $\imath = \sqrt{-1}$ is the imaginary unit. The corresponding eigenvalues of $\mathbb{A}$ are then given by

$$\lambda_p = \sum_{j=0}^{n-1}c_j\omega_p^j. \tag{24}$$

Define $\omega_m(s) = e^{\imath ms}$, then

$$\mathcal{H}_1\omega_m(s) = \begin{cases} \imath \mathrm{sign}(m)\omega_m(s), & m \in \mathbb{Z}^\star, \\ 0, & m = 0, \end{cases} \tag{25}$$

where $\mathbb{Z}^\star = \mathbb{Z} \setminus \{0\}$. Each 2π–periodic function φ has a Fourier expansion

$$\varphi(t) = \frac{1}{\sqrt{2\pi}} \sum_{m\in\mathbb{Z}} \hat{\varphi}(m)\omega_m(t), \tag{26}$$

where the Fourier coefficients are given by the formula

$$\hat{\varphi}(m) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \varphi(t)\bar{\omega}_m(t)\,dt.$$

Lemma 1. *The eigenvalues of* $\mathbb{A}$ *defined by* (9) and (18) *have the form*

$$\lambda_p = \begin{cases} 0, & p = 0, \\ \lambda_{p,r} + \imath\lambda_{p,i}, & p = 1, 2, \ldots, n-1 \end{cases} \tag{27}$$

if $\tau = -2/3$ *and*

$$\lambda_p = \begin{cases} 0, & p = 0, \\ -\lambda_{p,r} + \imath\lambda_{p,i}, & p = 1, 2, \ldots, n-1 \end{cases} \tag{28}$$

if $\tau = 2/3$, *where*

$$\begin{cases} \lambda_{p,r} = \frac{1}{\pi}\sin\frac{\pi p}{n} \sum_{k=0}^{\infty} (-1)^k \left(\frac{\sin(2\pi(k+p/n)/3)}{k+p/n} + \frac{\sin(2\pi(k+1-p/n)/3)}{k+1-p/n} \right), \\ \lambda_{p,i} = \frac{1}{\pi}\sin\frac{\pi p}{n} \sum_{k=0}^{\infty} (-1)^k \left(-\frac{\cos(2\pi(k+p/n)/3)}{k+p/n} + \frac{\cos(2\pi(k+1-p/n)/3)}{k+1-p/n} \right). \end{cases} \tag{29}$$

Proof. It suffices to discuss the case of $\tau = -2/3$, since the case of $\tau = 2/3$ can be considered by a same approach. By (25) and (26), we have

$$\mathcal{H}_1\varphi(s) = \frac{\imath}{\sqrt{2\pi}} \sum_{m\in\mathbb{Z}^\star} \mathrm{sign}(m)\hat{\varphi}(m)\omega_m(s).$$

Through direct calculation, for $0 \leqslant j \leqslant n-1$,

$$\begin{aligned} c_j &= \mathcal{H}_1\chi_j(s_0) = \frac{\imath}{\sqrt{2\pi}} \sum_{m\in\mathbb{Z}^\star} \mathrm{sign}(m)\hat{\chi}_j(m)\omega_m(h/6) = -\frac{1}{2\pi} \sum_{m\in\mathbb{Z}^\star} \frac{e^{\imath mh/6}}{|m|}\left(e^{-\imath m t_{j+1}} - e^{-\imath m t_j}\right) \\ &= -\frac{1}{2\pi}\sum_{m=1}^{\infty}\frac{1}{m}\left[\left(e^{-5\imath mh/6} - e^{\imath mh/6}\right)e^{-\imath mjh} + \left(e^{5\imath mh/6} - e^{-\imath mh/6}\right)e^{\imath mjh}\right] \\ &= \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{\sin(mh/2)}{m}\left[\sin\frac{mh}{3}\mathrm{Re}(e^{-\imath mjh}) - \cos\frac{mh}{3}\mathrm{Im}(e^{-\imath mjh})\right] \\ &= \frac{2}{\pi}\sum_{k=0}^{\infty}\sum_{l=1}^{n}\frac{\sin((kn+l)h/2)}{(kn+l)}\left[\sin\frac{(kn+l)h}{3}\mathrm{Re}(e^{-\imath ljh}) - \cos\frac{(kn+l)h}{3}\mathrm{Im}(e^{-\imath ljh})\right] \\ &= \frac{2}{\pi}\sum_{l=1}^{n-1}\sin\frac{lh}{2}\left[\mathrm{Re}(e^{-\imath ljh})\sum_{k=0}^{\infty}(-1)^k\frac{\sin((kn+l)h/3)}{kn+l} - \mathrm{Im}(e^{-\imath ljh})\sum_{k=0}^{\infty}(-1)^k\frac{\cos((kn+l)h/3)}{kn+l}\right]. \end{aligned}$$

Then, from (24), we have

$$\lambda_p = \frac{2}{\pi}\sum_{l=1}^{n-1}\sin\frac{lh}{2}\sum_{j=0}^{n-1}e^{-\imath pjh}\left[\mathrm{Re}(e^{-\imath ljh})\sum_{k=0}^{\infty}(-1)^k\frac{\sin((kn+l)h/3)}{kn+l} - \mathrm{Im}(e^{-\imath ljh})\sum_{k=0}^{\infty}(-1)^k\frac{\cos((kn+l)h/3)}{kn+l}\right],$$

which implies (27), by noting the fact that for $p + l \geqslant 0$,

$$2\sum_{j=0}^{n-1}e^{-\imath pjh}\mathrm{Re}(e^{-\imath ljh}) = \begin{cases} n, & l = p \quad \text{or } l = n-p, \\ 0, & \text{otherwise} \end{cases} \tag{30}$$

and

$$2\sum_{j=0}^{n-1} e^{-\imath pjh}\mathrm{Im}(e^{-\imath ljh}) = \begin{cases} n\imath, & l = p, \\ -n\imath, & l = n-p, \\ 0, & \text{otherwise}. \quad \Box \end{cases} \tag{31}$$

To estimate the bound of $\lambda_{p,r}$ and $\lambda_{p,i}$, we just need to discuss the functions

$$\begin{cases} \kappa_1(s) &= \frac{1}{\pi}\sin(\pi s)\sum_{k=0}^{\infty}(-1)^k\left(\frac{\sin(2\pi(k+s)/3)}{k+s} + \frac{\sin(2\pi(k+1-s)/3)}{k+1-s}\right), \\ \kappa_2(s) &= \frac{1}{\pi}\sin(\pi s)\sum_{k=0}^{\infty}(-1)^k\left(-\frac{\cos(2\pi(k+s)/3)}{k+s} + \frac{\cos(2\pi(k+1-s)/3)}{k+1-s}\right) \end{cases}$$

for $s \in (0,1)$.

Lemma 2. *$\kappa_1(s)$ and $\kappa_2(s)$ have the following properties:*

1 $\kappa_1(s)$ is symmetric about $1/2$, and $\kappa_2(s)$ is odd about $1/2$.
2 There exists two positive real constants C_1 and C_2 such that

$$\kappa_1(s) \geqslant C_1, \quad s \in [1/4, 1/2)$$

and

$$\kappa_2(s) \leqslant -C_2, \quad s \in (0, 1/4].$$

Proof. The proof of the first part is trivial. For the second part, let us define

$$\bar{\kappa}_1(s) = \sum_{k=0}^{\infty} a_k(s) + \sum_{k=0}^{\infty} b_k(s), \quad \bar{\kappa}_2(s) = \sum_{k=0}^{\infty} c_k(s)$$

with

$$a_k(s) = (-1)^k\frac{\sin(2\pi(k+s)/3)}{k+s}, \quad b_k(s) = (-1)^k\frac{\sin(2\pi(k+1-s)/3)}{k+1-s}$$

and

$$c_k(s) = (-1)^k\left(-\frac{\cos(2\pi(k+s)/3)}{k+s} + \frac{\cos(2\pi(k+1-s)/3)}{k+1-s}\right).$$

It can be easily verified that

$$\sum_{k=0}^{\infty} a_k(s) = \sum_{k=0}^{3} a_k(s) + \sum_{k=0}^{\infty}\sum_{l=1}^{6} a_{6k+l+3}(s), \quad \sum_{k=0}^{\infty} b_k(s) = \sum_{k=0}^{4} b_k(s) + \sum_{k=0}^{\infty}\sum_{l=1}^{6} b_{6k+l+4}(s)$$

and

$$\sum_{k=0}^{\infty} c_k(s) = \sum_{k=0}^{4} c_k(s) + \sum_{k=0}^{\infty}\sum_{l=1}^{6} c_{6k+l+4}(s).$$

Since for $k = 0, 1, \ldots$ and $0 < s \leqslant 1/2$,

$$\begin{aligned} \sum_{l=1}^{6} a_{6k+l+3}(s) &= \sin\frac{\pi(1-2s)}{3}\left(\frac{1}{6k+4+s} - \frac{1}{6k+7+s}\right) + \sin\frac{\pi(1+2s)}{3}\left(\frac{1}{6k+5+s} - \frac{1}{6k+8+s}\right) \\ &\quad + \sin\frac{2\pi s}{3}\left(\frac{1}{6k+6+s} - \frac{1}{6k+9+s}\right) \\ &\geqslant 0 \end{aligned}$$

and similarly

$$\sum_{l=1}^{6} b_{6k+l+4}(s) \geqslant 0, \quad \sum_{l=1}^{6} c_{6k+l+4}(s) \leqslant 0.$$

we get

$$\sum_{k=0}^{\infty} a_k(s) \geqslant \sum_{k=0}^{3} a_k(s), \quad \sum_{k=0}^{\infty} b_k(s) \geqslant \sum_{k=0}^{4} b_k(s)$$

and

$$\sum_{k=0}^{\infty} c_k(s) \leqslant \sum_{k=0}^{4} c_k(s),$$

Hence,

$$\bar{\kappa}_1(s) \geqslant \sum_{k=0}^{3} a_k(s) + \sum_{k=0}^{4} b_k(s) \triangleq \mu(s),$$

and

$$\bar{\kappa}_2(s) \leqslant \sum_{k=0}^{4} c_k(s)) \triangleq \nu(s),$$

with

$$\mu(s) = \sin\frac{2\pi s}{3}\left(\frac{1}{s} - \frac{1}{3-s} - \frac{1}{3+s}\right) + \sin\frac{\pi(1-2s)}{3}\left(\frac{1}{2-s} - \frac{1}{5-s} - \frac{1}{1+s}\right) + \sin\frac{\pi(1+2s)}{3}\left(\frac{1}{1-s} - \frac{1}{4-s} - \frac{1}{2+s}\right)$$

and

$$\nu(s) = -\cos\frac{2\pi s}{3}\left(\frac{1}{s} - \frac{1}{3-s} - \frac{1}{3+s}\right) + \cos\frac{\pi(1-2s)}{3}\left(\frac{1}{2-s} - \frac{1}{5-s} - \frac{1}{1+s} + \frac{1}{4+s}\right)$$
$$- \cos\frac{\pi(1+2s)}{3}\left(\frac{1}{1-s} - \frac{1}{4-s} - \frac{1}{2+s}\right).$$

From Fig. 1, we see that $\mu(s)$ is positive and monotonic increasing and $\nu(s)$ is negative and monotonic increasing for $s \in (0, 1/2]$. Therefore,

$$\kappa_1(s) \geqslant \frac{\sqrt{2}}{2\pi}\mu(1/4) \approx 0.4687$$

for $s \in (1/4, 1/2]$, and

$$\kappa_2(s) \leqslant \frac{\sqrt{2}}{2\pi}\nu(1/4) \approx -0.6886$$

for $s \in (0, 1/4]$. □

From Lemma 1 and Lemma 2, we can easily obtain the following result.

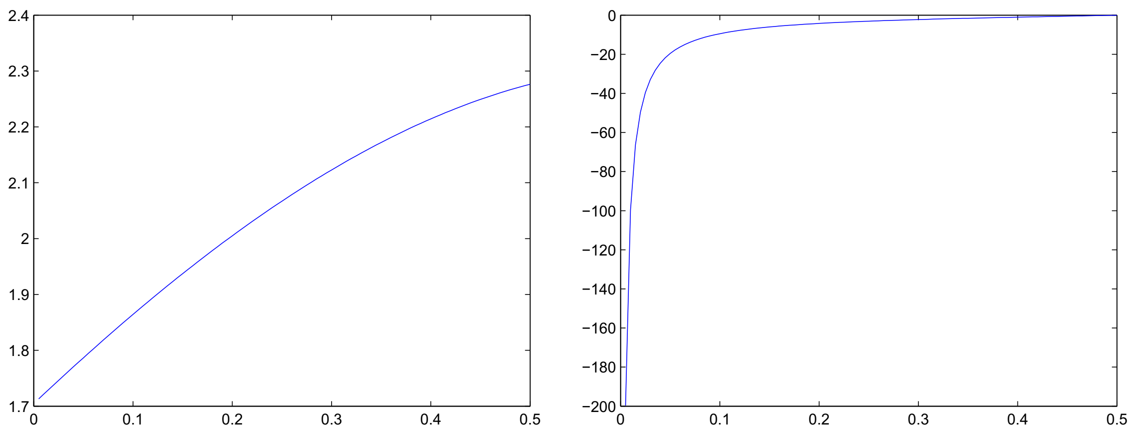

Fig. 1. $\mu(s)$ and $\nu(s)$.

Lemma 3. *If* $\tau = \pm 2/3$, *then*

$$|\lambda_p|^2 = \lambda_{p,r}^2 + \lambda_{p,i}^2 \geqslant C_0^2 \tag{32}$$

for $p = 1, 2, \ldots, n-1$, *where* $C_0 = \min\{C_1, C_2\}$ *and* C_1, C_2 *is defined in Lemma 2.*

Theorem 4. *The eigenvalues of* $\bar{\mathbb{A}}$ *defined by (12) are* $\pm\sqrt{n}$ *and* $\lambda_p, \quad p = 1, 2, \ldots, n-1$ *and the corresponding eigenvectors are*

$$\bar{\mathbf{v}}_0^{(1,2)} = (2n)^{-1/2}(\pm\sqrt{n}, \underbrace{1, 1, \ldots, 1_n})^T$$

and

$$\bar{\mathbf{v}}_p = \begin{pmatrix} 0 \\ \mathbf{v}_p \end{pmatrix}, \quad p = 1, 2, \ldots, n-1,$$

where $\mathbf{v}_p$ *is defined by (23). Moreover,* $\bar{\mathbf{v}}_0^{(1,2)}$ *and* $\bar{\mathbf{v}}_p, p = 1, 2, \ldots, n-1$ *are orthogonal to each other, and hence*

$$\|\bar{\mathbb{A}}^{-1}\|_2 \leqslant C_0^{-1} \tag{33}$$

for sufficiently large n.

Proof. It is easy to verify that

$$\bar{\mathbb{A}}\bar{\mathbf{v}}_0^{(1)} = \sqrt{n}\bar{\mathbf{v}}_0^{(1)}$$

and

$$\bar{\mathbb{A}}\bar{\mathbf{v}}_0^{(2)} = -\sqrt{n}\bar{\mathbf{v}}_0^{(2)}.$$

Moreover, for $p = 1, 2, \ldots, n-1$,

$$\bar{\mathbb{A}}\bar{\mathbf{v}}_p = \begin{pmatrix} 0 & \mathbf{e}_n^T \\ \mathbf{e}_n & \mathbb{A} \end{pmatrix}\begin{pmatrix} 0 \\ \mathbf{v}_p \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbb{A}\mathbf{v}_p \end{pmatrix} = \lambda_p \begin{pmatrix} 0 \\ \mathbf{v}_p \end{pmatrix} = \lambda_p \bar{\mathbf{v}}_p.$$

When n is sufficiently large,

$$\|\bar{\mathbb{A}}^{-1}\|_2 = \sqrt{\max_{1 \leqslant p \leqslant n-1}\{1/n, |\lambda_p^{-1}|^2\}} \leqslant C_0^{-1}. \qquad \square$$

Theorem 5. *Assume that* $\varphi(t)$, *the solution of Cauchy singular integral equation (1) with kernel* $\kappa(t) = \frac{1}{2\pi}\cot\frac{t}{2}$, *belongs to* $C^3[0, 2\pi]$. *Then, for the collocation scheme (10) with (18) and collocation points* $s_i = t_i + (1+\tau)h/2 (0 \leqslant i \leqslant n-1)$ *with* $\tau = -2/3$ or $2/3$, *there holds the error estimate*

$$\left(\sum_{i=0}^{n-1}|\varphi(\hat{t}_i) - \varphi_i|^2 h\right)^{1/2} \leqslant Ch^2. \tag{34}$$

Proof. Since $\sum_{j=0}^{n-1}\varphi(\hat{t}_j)h$ is actually the midpoint rule of the integral $\int_0^{2\pi}\varphi(t)\,dt$, from the assumption $\varphi \in C^3[0, 2\pi]$, we see that

$$\left|\sum_{j=0}^{n-1}\varphi(\hat{t}_j)h\right| = \left|\int_0^{2\pi}\varphi(t)\,dt - \sum_{j=0}^{n-1}\varphi(\hat{t}_j)h\right| \leqslant Ch^3.$$

Combining with Theorem 1, Theorem 3 and (33) yields (34). $\square$

3.2. Trapezoidal rule and its related collocation scheme

Substituting φ in (16) with its piecewise linear interpolation $\Pi_h\varphi$ defined by (2) with (4) yields the composite trapezoidal rule

$$\mathcal{Q}_h\varphi(s) = \frac{1}{2\pi}\int_0^{2\pi}\Pi_h\varphi(t)\cot\frac{t-s}{2}\,dt = \sum_{j=0}^{n-1}\varpi_j(s)\varphi(t_j), \tag{35}$$

where

$$\varpi_j(s)=\frac{1}{\pi h}\left[(s-t_{j-1})\ln\left|\frac{\sin((t_j-s)/2)}{\sin(t_{j-1}-s)/2}\right|+(t_{j+1}-s)\ln\left|\frac{\sin((t_{j+1}-s)/2)}{\sin(t_j-s)/2}\right|\right]+\frac{1}{2\pi h}[2\mathrm{J}_1(t_j-s)-\mathrm{J}_1(t_{j-1}-s)-\mathrm{J}_1(t_{j+1}-s)] \quad (36)$$

for $j=0,1,\ldots,n-1$ with $t_{-1}+2\pi=t_{n-1}$, and

$$\mathrm{J}_1(x)=\int_0^x t\cot\frac{t}{2}dt=2x\ln\left(2\sin\frac{x}{2}\right)+2\mathrm{Cl}_2(x), \quad (37)$$

where

$$\mathrm{Cl}_2(x)=-\int_0^x \ln\left|2\sin\frac{t}{2}\right|dt$$

is the Clausen integral [18]. The error estimate of trapezoidal rule $\mathcal{Q}_h\varphi$ for $\mathcal{H}_1\varphi(s)$ defined by (16) can be stated in the following theorem.

Theorem 6. *Assume that $\varphi\in C^2[0,2\pi]$ and $s\neq t_i$ for any $i=0,\ldots,n$. Then, for the composite trapezoidal rule $\mathcal{Q}_h\varphi$ defined in (35) and (36), there exists a positive constant C, independent of h and s, such that*

$$|\,\mathcal{E}\varphi(s)\,|=|\,\mathcal{H}_1\varphi(s)-\mathcal{Q}_h\varphi(s)\,|\leqslant C(|\ln\gamma(h,s)|+|\ln h|)h^2. \quad (38)$$

where $\gamma(h,s)$ is defined by (21).

Proof. Define $\mathcal{E}(t)=\varphi(t)-\Pi_h\varphi(t)$ and

$$\tilde{\kappa}_s(t)=\begin{cases}(t-s)\cot\frac{t-s}{2}, & t\neq s\\ 2, & t=s.\end{cases} \quad (39)$$

Then, from (35) and (39), we get

$$\mathcal{E}\varphi(s)=\int_0^{2\pi}\mathcal{E}(t)\cot\frac{t-s}{2}dt=\int_0^{2\pi}\frac{\mathcal{E}(t)\tilde{\kappa}_s(t)}{t-s}dt=2\int_0^{2\pi}\frac{\mathcal{E}(t)}{t-s}dt+\int_0^{2\pi}\frac{\mathcal{E}(t)[\tilde{\kappa}_s(t)-2]}{t-s}dt.$$

From Theorem 2.1 in [9], it follows

$$\left|\int_0^{2\pi}\frac{\mathcal{E}(t)}{t-s}dt\right|\leqslant C|\ln h|h^2. \quad (40)$$

As for the second part, we split it into two parts

$$\int_0^{2\pi}\frac{\mathcal{E}(t)[\tilde{\kappa}_s(t)-2]}{t-s}dt=\int_{t_m}^{t_{m+1}}\frac{\mathcal{E}(t)[\tilde{\kappa}_s(t)-2]}{t-s}dt+\sum_{\substack{i=0\\ i\neq m}}^{n-1}\int_{t_i}^{t_{i+1}}\frac{\mathcal{E}(t)[\tilde{\kappa}_s(t)-2]}{t-s}dt.$$

By using the same approach used in [8], we get

$$\left|\int_{t_m}^{t_{m+1}}\frac{\mathcal{E}(t)[\tilde{\kappa}_s(t)-2]}{t-s}dt\right|\leqslant C\,|\ln\gamma(h,s)|\,h^2,$$

which leads to

$$\begin{aligned}\left|\int_0^{2\pi}\frac{\mathcal{E}(t)[\tilde{\kappa}_s(t)-2]}{t-s}dt\right|&\leqslant C|\ln\gamma(h,s)|h^2+\max_{t\in[0,2\pi]}\{|\mathcal{E}(t)[\tilde{\kappa}_s(t)-2]|\}\sum_{\substack{i=0\\ i\neq m}}^{n-1}\int_{t_i}^{t_{i+1}}\frac{1}{|t-s|}dt\\ &\leqslant C|\ln\gamma(h,s)|h^2+Ch^2\ln\frac{(2\pi-s)s}{(t_{m+1}-s)(s-t_m)}\leqslant C(|\ln\gamma(h,s)|+|\ln h|)h^2.\end{aligned} \quad (41)$$

Putting (40) and (41) together we can get (38). □

In this case, we can choose the midpoints as the collocation points.

Lemma 4. *The eigenvalues of $\mathbb{A}$ defined by (9) and (36) with $\tau=0$ can be represented as*

$$\lambda_p=\begin{cases}0, & p=0,\\ \frac{1}{2\pi^2}\sin\frac{\pi p}{n}\sum_{k=0}^{\infty}(-1)^k\left[\frac{1}{(k+p/n)^2}+\frac{1}{(k+1-p/n)^2}\right](e^{-\imath ph}-1), & p=1,2,\ldots,n-1.\end{cases} \quad (42)$$

Proof. By (25), for $0 \leqslant j \leqslant n-1$, we have

$$c_j = \mathcal{H}_1\phi_j(s_0) = \frac{1}{2\pi h}\sum_{m\in\mathbb{Z}^\star}\frac{\text{sign}(m)}{m}e^{\imath mh/2}\left[\int_{t_{j-1}}^{t_j}e^{-\imath mt}dt - \int_{t_j}^{t_{j+1}}e^{-\imath mt}dt\right] = \frac{2}{\pi h}\sum_{m=1}^{\infty}\frac{\sin(mh/2)}{m^2}\text{Re}[e^{-\imath m(j-1)h} - e^{-\imath mjh}]$$
$$= \frac{2}{\pi h}\sum_{k=0}^{\infty}\sum_{l=1}^{n}\frac{\sin((kn+l)h)/2)}{(kn+l)^2}\text{Re}[e^{-\imath(kn+l)(j-1)h} - e^{-\imath(kn+l)jh}] = \frac{2}{\pi h}\sum_{l=1}^{n-1}\sin\frac{lh}{2}\text{Re}[e^{-\imath l(j-1)h} - e^{-\imath ljh}]\sum_{k=0}^{\infty}\frac{(-1)^k}{(kn+l)^2}.$$

Then, from (24),

$$\lambda_p = \frac{2}{\pi h}\sum_{l=1}^{n-1}\sin\frac{lh}{2}\sum_{j=0}^{n-1}e^{-\imath pjh}\text{Re}[e^{-\imath l(j-1)h} - e^{-\imath ljh}]\sum_{k=0}^{\infty}\frac{(-1)^k}{(kn+l)^2},$$

which implies (42), by noting the fact that for $p+l \geqslant 0$,

$$2\sum_{j=0}^{n-1}e^{-\imath pjh}\text{Re}(e^{-\imath jlh}) = \begin{cases} n, & l=p \quad \text{or } l=n-p, \\ 0, & \text{otherwise} \end{cases}$$

and

$$2\sum_{j=0}^{n-1}e^{-\imath pjh}\text{Re}(e^{-\imath(j-1)lh}) = \begin{cases} ne^{-\imath ph}, & l=p \text{ or } l=n-p, \\ 0, & \text{otherwise}, \end{cases} \qquad \square$$

Theorem 7. *The eigenvalues of* $\bar{\mathbb{A}}$ *defined by* (12) *are* $\pm\sqrt{n}$ *and* $\lambda_p, p = 1,2,\ldots,n-1$, *and there exists a positive constant C, such that*

$$\|\bar{\mathbb{A}}^{-1}\|_2 \leqslant C$$

for sufficiently large n.

Proof. The first part of the statements is trivial. From (42), we have

$$|\lambda_p| = \frac{1}{\pi^2}\sin^2\frac{p\pi}{n}\sum_{k=0}^{\infty}(-1)^k\left[\frac{1}{(k+p/n)^2} + \frac{1}{(k+1-p/n)^2}\right]$$

for $p = 1,2,\ldots,n-1$. Let us define

$$\zeta(s) = \frac{1}{\pi^2}\sin^2\pi s\sum_{k=0,1}(-1)^k\left[\frac{1}{(k+s)^2} + \frac{1}{(k+1-s)^2}\right], \quad s \in (0,1).$$

It's easy to verify that $\zeta(s)$ is symmetric about $1/2$, positive and monotonic decreasing for $s \in (0,1/2]$. Hence

$$|\lambda_p| \geqslant \zeta(1/2) \approx 0.7205,$$

which implies that for sufficiently large n

$$\|\bar{\mathbb{A}}^{-1}\|_2 \leqslant \sqrt{\max_{1\leqslant p\leqslant n-1}\{1/n, |\lambda_p^{-1}|^2\}} \leqslant C. \qquad \square$$

The following theorem gives the error estimate of (10) based on the trapezoidal rule, and we omit the proof here since it is similar to that of Theorem 5.

Theorem 8. *Assume that* $\varphi(t)$, *the solution of Cauchy singular integral equation* (1) *with kernel* $\kappa(t) = \frac{1}{2\pi}\cot\frac{t}{2}$, *belongs to* $C^3[0,2\pi]$. *Then, for the collocation scheme* (10) *with* (36) *and collocation points* $s_i = t_i + (1+\tau)h/2 (0 \leqslant i \leqslant n-1)$ *with* $\tau = 0$, *there holds the error estimate*

$$\left(\sum_{i=0}^{n-1}|\varphi(t_i) - \varphi_i|^2 h\right)^{1/2} \leqslant Ch^2|\ln h|. \tag{43}$$

Remark 3. It should be mentioned that the collocation scheme discussed in this subsection is no other than that suggested in [3], where only a suboptimal convergence rate $\mathcal{O}(h|\ln h|)$ in maximum norm was established by using a discrete maximum principle.

4. Case of $\kappa(t) = \frac{1}{4\pi \sin^2(t/2)}$

In this section, we consider the hypersingular integral equation.

$$\mathcal{H}_2\varphi(s) := \frac{1}{4\pi} \oint_0^{2\pi} \frac{\varphi(t)}{\sin^2 \frac{t-s}{2}} dt = g(s), \quad s \in (0, 2\pi) \tag{44}$$

with the compatibility condition

$$\int_0^{2\pi} g(t)\, dt = 0. \tag{45}$$

Hypersingular integral in (44) must be understood in the Hadamard sense. As in [14], we see that under the condition (45), there exists a unique solution up to an additive constant for the integral equation (44). In order to arrive at a unique solution, we adopt a periodical condition $\int_0^{2\pi} \varphi(t)\, dt = 0$.

4.1. Midpoint rule and its related collocation scheme

Substituting φ in $\mathcal{H}_2\varphi(s)$ with its piecewise constant interpolation defined by (2) with (3) $\Pi_h\varphi$ yields the composite midpoint rule.

$$\mathcal{Q}_h\varphi(s) = \frac{1}{4\pi} \oint_0^{2\pi} \frac{\Pi_h\varphi(t)}{\sin^2 \frac{t-s}{2}} dt = \sum_{j=0}^{n-1} \varpi_j(s)\varphi(\hat{t}_j), \tag{46}$$

where

$$\varpi_j(s) = \frac{1}{2\pi}\left(\cot\frac{t_j - s}{2} - \cot\frac{t_{j+1} - s}{2}\right). \tag{47}$$

Theorem 9. [5] *Assume that $\mathcal{Q}_h\varphi$ be computed by (46) and (47) with a uniform mesh. Then, for $\varphi \in C^{3+\alpha}[0, 2\pi](0 < \alpha < 1)$ at $s^\star = \hat{t}_m (0 \leqslant m \leqslant n-1)$, we have*

$$|\mathcal{E}\varphi(s^\star)| = |\mathcal{H}_2\varphi(s^\star) - \mathcal{Q}_h\varphi(s^\star)| \leqslant Ch^2.$$

In this case, we have no other choice but to take the midpoints as the collocation points, since the midpoint rule $\mathcal{Q}_h\varphi$ defined by (46) and (47) is divergent in general (see Remark 2.1 in [5]).

Lemma 5. *The eigenvalues of $\mathbb{A}$ defined by (9) and (47) with $\tau = 0$ have the form*

$$\lambda_p = \begin{cases} 0, & p = 0, \\ -\frac{n}{\pi} \sin\frac{\pi p}{n}, & p = 1, 2, \ldots, n-1. \end{cases} \tag{48}$$

Proof. Use the identity

$$\cot\frac{(j \pm 1/2)h}{2} = \sum_{k=1}^{n} \sin k(j \pm 1/2)h$$

and through direct calculation

$$c_j = \frac{1}{2\pi}\left[\cot\frac{(j-1/2)h}{2} - \cot\frac{(j+1/2)h}{2}\right] = \frac{1}{2\pi}\sum_{k=1}^{n}[\sin k(j-1/2)h - \sin k(j+1/2)h] = -\frac{1}{\pi}\sum_{k=1}^{n} \sin\frac{kh}{2} \cos kjh,$$
$$0 \leqslant j \leqslant n-1. \tag{49}$$

From (24), we have.

$$\lambda_p = \sum_{j=0}^{n-1} c_j e^{-\imath pjh} = -\frac{1}{\pi}\sum_{k=1}^{n} \sin\frac{kh}{2} \sum_{j=0}^{n-1} e^{-\imath pjh} \cos kjh. \tag{50}$$

which implies (48) by noting the fact that for $k + p \geqslant 1$,

$$2\sum_{j=0}^{n-1} e^{-\imath pjh} \cos kjh = \begin{cases} n, & k = p \text{or} k = n-p, \\ 0, & \text{otherwise}. \quad \square \end{cases} \tag{51}$$

Theorem 10. *The eigenvalues of $\bar{\mathbb{A}}$ defined by (12) are $\pm\sqrt{n}$ and $\lambda_p, p = 1, 2, \ldots, n-1$, and hence*

$$\|\bar{\mathbb{A}}^{-1}\|_2 \leqslant C$$

for sufficiently large n.

Proof. The first part of the theorem can be easily verified. From (48) we know that $\lambda_p < 0$ for all p and $\lambda_p = \lambda_{n-p}$, and thus λ_p is symmetric about $n/2$. Furthermore, λ_p is monotonic decreasing for $p \in [1, n/2]$, which leads to

$$|\lambda_p| \geqslant |\lambda_1| = \frac{2}{h}\sin\frac{h}{2} \to 1 \text{ as } h \to 0.$$

This means that if h is small enough, then $|\lambda_p|$ is greater than $1/2$. Therefore, for sufficiently large n,

$$\|\bar{\mathbb{A}}^{-1}\|_2 = \sqrt{\max_{1\leqslant p\leqslant n-1}\{1/n, \lambda_p^{-2}\}} \leqslant C. \qquad \square$$

Theorem 11. *Assume that $\varphi(t)$, the solution of hypersingular integral equation (1) with kernel $\kappa(t) = \frac{1}{4\pi\sin^2(t/2)}$, belongs to $C^{3+\alpha}[0, 2\pi](0 < \alpha < 1)$. Then, for the collocation scheme (10) with (47) and collocation points $s_i = t_i + h/2 (0 \leqslant i \leqslant n-1)$, there holds the error estimate*

$$\left(\sum_{i=0}^{n-1}|\varphi(\hat{t}_i) - \varphi_i|^2 h\right)^{1/2} \leqslant Ch^2.$$

The proof can be finished by the same approach in Theorem 5.

Remark 4. This scheme has been discussed in a quite different way in [5], where the coefficient matrix of the scheme has an explicit inversion expression, by which an optimal convergence rate in maximum norm was established. Here we get an optimal L_2 error estimate in a more simple way.

4.2. Trapezoidal rule and its related collocation scheme

Substituting φ in $\mathcal{H}_2\varphi(s)$ with its piecewise linear interpolation $\Pi_h\varphi$ defined by (2) with (4) yields the composite trapezoidal rule.

$$\mathcal{Q}_h\varphi(s) = \frac{1}{4\pi}\,\text{=}\!\!\!\!\!\!\int_0^{2\pi}\frac{\Pi_h\varphi(t)}{\sin^2\frac{t-s}{2}}dt = \sum_{j=0}^{n-1}\varpi_j(s)\varphi(t_j), \tag{52}$$

where

$$\varpi_j(s) = \frac{1}{\pi h}\ln\left|\frac{1-\cos(t_j - s)}{\cos h - \cos(t_j - s)}\right|. \tag{53}$$

Theorem 12. [19] *Assume that $\mathcal{Q}_h\varphi$ be computed by (52) and (53) with a uniform mesh. Then, for $\varphi \in C^{3+\alpha}[0, 2\pi](0 < \alpha < 1)$ at $s^\star = t_m + (1+\tau^\star)h/2 (0 \leqslant m \leqslant n-1)$ with $\tau^\star = \pm 2/3$, we have*

$$|\mathcal{E}\varphi(s^\star)| = |\mathcal{H}_2\varphi(s^\star) - \mathcal{Q}_h\varphi(s^\star)| \leqslant Ch^2. \tag{54}$$

In this case, we can choose any one of two series of points $\{s_i = t_i + (1+\tau)h/2\}$ with $\tau = 2/3$ or $-2/3$ as the collocation points.

Lemma 6. *The eigenvalues of $\mathbb{A}$ defined by (9) and (53) have the form*

$$\lambda_p = \begin{cases} 0, & p = 0, \\ (\lambda_{p,r} + \imath\lambda_{p,i})(e^{-\imath ph} - 1)/h, & p = 1, 2, \ldots, n-1 \end{cases} \tag{55}$$

for $\tau = -2/3$ and

$$\lambda_p = \begin{cases} 0, & p = 0, \\ (-\lambda_{p,r} + \imath\lambda_{p,i})(e^{-\imath ph} - 1)/h, & p = 1, 2, \ldots, n-1 \end{cases} \tag{56}$$

for $\tau = 2/3$, where $\lambda_{p,r}$ and $\lambda_{p,i}$ are defined by (29).

Proof. We only consider the case of $\tau = -2/3$, since the case of $\tau = 2/3$ can be obtained by a same approach. Since

$$\mathcal{H}_2\omega_m(s) = \begin{cases} -m\text{sign}(m)\omega_m(s), & m \in \mathbb{Z}^\star, \\ 0, & m = 0, \end{cases}$$

then, from (26) we have

$$\mathcal{H}_2\varphi(s) = -\frac{1}{\sqrt{2\pi}}\sum_{m\in\mathbb{Z}^\star} m\text{sign}(m)\hat{\varphi}(m)\omega_m(s). \tag{57}$$

Through direct calculation, for $0 \leqslant j \leqslant n-1$,

$$c_j = \mathcal{H}_2\phi_j(s_0) = \frac{\imath}{2\pi h}\sum_{m\in\mathbb{Z}^\star}\text{sign}(m)e^{\imath mh/6}\left[\int_{t_{j-1}}^{t_j} e^{-\imath mt}dt - \int_{t_j}^{t_{j+1}} e^{-\imath mt}dt\right].$$

As the proof of Lemma 1, we can prove

$$\sum_{m\in\mathbb{Z}^\star}\text{sign}(m)e^{\imath mh/6}\int_{t_j}^{t_{j+1}} e^{-\imath mt}dt = -4\imath\sum_{l=1}^{n-1}\sin\frac{lh}{2}\left[\text{Re}(e^{-\imath ljh})\sum_{k=0}^{\infty}(-1)^k\frac{\sin((kn+l)h/3)}{kn+l} - \text{Im}(e^{-\imath ljh})\sum_{k=0}^{\infty}(-1)^k\frac{\cos((kn+l)h/3)}{kn+l}\right]$$

and

$$\sum_{m\in\mathbb{Z}^\star}\text{sign}(m)e^{\imath mh/6}\int_{t_{j-1}}^{t_j} e^{-\imath mt}dt = -4\imath\sum_{l=1}^{n-1}\sin\frac{lh}{2}\left[\text{Re}(e^{-\imath l(j-1)h})\sum_{k=0}^{\infty}(-1)^k\frac{\sin((kn+l)h/3)}{kn+l} - \text{Im}(e^{-\imath l(j-1)h})\sum_{k=0}^{\infty}(-1)^k\frac{\cos((kn+l)h/3)}{kn+l}\right].$$

Then, from (24), we have

$$\lambda_p = \frac{2}{\pi h}\sum_{l=1}^{n-1}\sin\frac{lh}{2}\left[\sum_{j=0}^{n-1}e^{-\imath pjh}\text{Re}(e^{-\imath l(j-1)h} - e^{-\imath ljh})\sum_{k=0}^{\infty}(-1)^k\frac{\sin((kn+l)h/3)}{kn+l}\right.$$
$$\left.-\sum_{j=0}^{n-1}e^{-\imath pjh}\text{Im}(e^{-\imath l(j-1)h} - e^{-\imath ljh})\sum_{k=0}^{\infty}(-1)^k\frac{\cos((kn+l)h/3)}{kn+l}\right],$$

which implies (55), by noting the fact that for $p+l \geqslant 0$,

$$2\sum_{j=0}^{n-1}e^{-\imath pjh}\text{Re}(e^{-\imath l(j-1)h} - e^{-\imath ljh}) = \begin{cases} n(e^{-\imath ph}-1), & l = p \text{ or } l = n-p, \\ 0, & \text{otherwise}, \end{cases}$$

$$2\sum_{j=0}^{n-1}e^{-\imath pjh}\text{Im}(e^{-\imath l(j-1)h} - e^{-\imath ljh}) = \begin{cases} n\imath(e^{-\imath ph}-1), & l = p, \\ -n\imath(e^{-\imath ph}-1), & l = n-p, \\ 0, & \text{otherwise}. \end{cases} \qquad \square$$

Theorem 13. *The eigenvalues of $\bar{\mathbb{A}}$ defined by (12) are $\pm\sqrt{n}$ and $\lambda_p, \quad p = 1,2,\ldots,n-1$. Hence, there exists a positive constant C, such that*

$$\|\bar{\mathbb{A}}^{-1}\|_2 \leqslant C$$

for sufficiently large n.

Proof. The first half of the theorem can be easily verified. From (55) and (56) we have

$$|\lambda_p| = \sqrt{\lambda_{p,r}^2 + \lambda_{p,i}^2}\left|\frac{e^{-\imath ph}-1}{h}\right|, \quad p = 1,2,\ldots,n-1. \tag{58}$$

By Lemma 3 we see that $\sqrt{\lambda_{p,r}^2 + \lambda_{p,i}^2} \geqslant C_0$. In addition,

$$\left|\frac{e^{-\imath ph}-1}{h}\right| = \frac{2\sin(ph/2)}{h} \geqslant \frac{2\sin(h/2)}{h} \to 1 \text{ as } h \to 0,$$

which means that if h is sufficiently small, $\left|\frac{e^{-\imath ph}-1}{h}\right|$ is greater than $1/2$. Hence,

$$\|\bar{\mathbb{A}}^{-1}\|_2 = \sqrt{\max_{1\leqslant p\leqslant n-1}\{1/n, |\lambda_p^{-1}|^2\}} \leqslant C. \qquad \square$$

We conclude this section by the following Theorem, whose proof is also similar to that of Theorem 5.

Theorem 14. *Assume that $\varphi(t)$, the solution of the hypersingular integral equation of first kind with kernel $\kappa(t) = \frac{1}{4\pi \sin^2(t/2)}$, belongs to $C^{3+\alpha}[0, 2\pi](0 < \alpha < 1)$. Then, for the collocation scheme (10) with (53) and collocation points $s_i = t_i + (1+\tau)h/2$ $(0 \leqslant i \leqslant n-1)$ with $\tau = -2/3$ or $2/3$, there holds the error estimate*

$$\left(\sum_{i=0}^{n-1} |\varphi(t_i) - \varphi_i|^2 h\right)^{1/2} \leqslant Ch^2.$$

5. The proof of Theorem 2

Before approaching the proof of Theorem 2, let us introduce some lemmas and additional notations. Define

$$P_k(t) = \prod_{j=0}^{k} (t - t_j) = \prod_{j=0}^{k} \left(t - \frac{2j-k}{k}\right)$$

and denote by Q_k the function of second kind associated with P_k, defined by

$$Q_k(x) = \begin{cases} \frac{1}{2} ⨍_{-1}^{1} \frac{P_k(t)}{x-t} dt, |x| < 1, \\ \frac{1}{2} \int_{-1}^{1} \frac{P_k(t)}{x-t} dt, |x| > 1. \end{cases}$$

Lemma 7. *Assume the singular point $s \in (t_m, t_{m+1})$, and let $c_i = 2(s - t_i)/h - 1$, then*

$$Q_1(c_i) = \begin{cases} -\frac{1}{h} ⨍_{t_i}^{t_{i+1}} \frac{t-\hat{t}_i}{t-s} dt, & i = m, \\ -\frac{1}{h} \int_{t_i}^{t_{i+1}} \frac{t-\hat{t}_i}{t-s} dt, & i \neq m. \end{cases}$$

Proof. By the definition of $Q_1(x)$ and variable substitution, we have

$$⨍_{t_m}^{t_{m+1}} \frac{t-\hat{t}_i}{t-s} dt = \lim_{\varepsilon \to 0} \left\{ \int_{t_m}^{s-\varepsilon} \frac{t-\hat{t}_m}{t-s} dt + \int_{s-\varepsilon}^{t_m} \frac{t-\hat{t}_m}{t-s} dt \right\} = \frac{h}{2} \lim_{\varepsilon \to 0} \left\{ \int_{-1}^{c_m - 2\varepsilon/h} \frac{x}{x-c_m} dx + \int_{c_m+2\varepsilon/h}^{1} \frac{x}{x-c_m} dx \right\}$$
$$= \frac{h}{2} ⨍_{-1}^{1} \frac{x}{x-c_m} dx = -hQ_1(c_m),$$

where c_i's equivalent form $c_i = 2(m-i) + \tau$ has been used. The case $i \neq m$ can be proved similarly. □

Define the operator $\mathcal{L}$ by

$$\mathcal{L}(\psi; \tau) = \psi(\tau) + \sum_{i=1}^{\infty} [\psi(2i+\tau) + \psi(-2i+\tau)], \tau \in (-1, 1). \tag{59}$$

Obviously, $\mathcal{L}$ is a linear operator on ψ.

Lemma 8 [19]. *Let $\mathcal{L}$ be a linear operator defined in (59), we have*

$$\mathcal{L}(Q_1; \tau) = -\ln\left(2\cos\frac{\tau\pi}{2}\right), \quad |\tau| < 1.$$

Lemma 9. *Let $\mathcal{I}_{n,i}(s)$ be defined by*

$$\mathcal{I}_{n,i}(s) = \begin{cases} ⨍_{t_m}^{t_{m+1}} (t - \hat{t}_i) \cot\frac{t-s}{2} dt, & i = m, \\ \int_{t_i}^{t_{i+1}} (t - \hat{t}_i) \cot\frac{t-s}{2} dt, & i \neq m, \end{cases} \tag{60}$$

then

$$\sum_{i=0}^{n-1} \mathcal{I}_{n,i}(s) = 2h \ln\left(2\cos\frac{\tau\pi}{2}\right).$$

Proof. Using the equality [1]

$$\cot \pi x = \frac{1}{\pi x} + \frac{1}{\pi}\sum_{\substack{l\neq 0\\ l=-\infty}}^{\infty}\frac{1}{x+l},$$

we can easily get

$$\cot\frac{t-s}{2} = \frac{2}{t-s} + \sum_{\substack{l=-\infty\\ l\neq 0}}^{\infty}\frac{2}{t-s+2l\pi}.$$

From Lemma 7 and Lemma 8, we have

$$\begin{aligned}\sum_{i=0}^{n-1}\mathcal{I}_{n,i}(s) &= \int_{t_m}^{t_{m+1}}(t-\hat{t}_i)\cot\frac{t-s}{2}dt + \sum_{\substack{i=0\\ i\neq m}}^{n-1}\int_{t_i}^{t_{i+1}}(t-\hat{t}_i)\cot\frac{t-s}{2}dt \\ &= 2\int_{t_m}^{t_{m+1}}\frac{t-\hat{t}_i}{t-s}dt + 2\sum_{\substack{i=0\\ i\neq m}}^{n-1}\int_{t_i}^{t_{i+1}}\frac{t-\hat{t}_i}{t-s}dt + 2\sum_{\substack{i=0\\ i\neq m}}^{n-1}\sum_{\substack{l=-\infty\\ l\neq 0}}^{\infty}\int_{t_i}^{t_{i+1}}\frac{t-\hat{t}_i}{t-s+2l\pi}dt = -2h\sum_{i=0}^{n-1}\sum_{l=-\infty}^{\infty}Q_1(2(m-i+nl)+\tau) \\ &= -2h\mathcal{L}(Q_1,\tau) = 2h\ln\left(2\cos\frac{\tau\pi}{2}\right). \qquad \square\end{aligned}$$

Lemma 10. *Let $\mathcal{I}_{n,i}(s)$ be defined in (60), then for $\varphi \in C^{2+\alpha}[0,2\pi](0<\alpha<1)$, we have*

$$\sum_{\substack{i=0\\ i\neq m}}^{n-1}[\varphi'(\eta_i)-\varphi'(s)]\mathcal{I}_{n,i}(s) \leqslant C\gamma^{-2}(h,s)\rho(s)h^2, \tag{61}$$

where $\eta_i \in [t_i, t_{i+1}]$.

Proof. If $i \neq m$, integrating by parts on $\mathcal{I}_{n,i}(s)$ leads to

$$\mathcal{I}_{n,i}(s) = h\left(\ln\left|\sin\frac{t_i-s}{2}\right| + \ln\left|\sin\frac{t_{i+1}-s}{2}\right|\right) - 2\int_{t_i}^{t_{i+1}}\ln\left|\sin\frac{t-s}{2}\right|,$$

which is actually the error of the trapezoidal rule for certain Riemann integral on $[t_i,t_{i+1}]$. Therefore, there exists $\tilde{t}_i \in [t_i,t_{i+1}]$ such that

$$\mathcal{I}_{n,i}(s) = \frac{h^3}{24\sin^2\frac{\tilde{t}_i-s}{2}}.$$

Since $\varphi(t)\in C^{2+\alpha}[0,2\pi]$ $(0<\alpha<1)$, we have

$$\left|\sum_{\substack{i=0\\ i\neq m}}^{n-1}[\varphi'(\eta_i)-\varphi'(s)]\mathcal{I}_{n,i}(s)\right| \leqslant C\rho(s)\left[\sum_{i=0}^{m-1}\frac{h^3(s-t_i)^{1+\alpha}}{24(s-t_{i+1})^2} + \sum_{i=m+1}^{n-1}\frac{h^3(t_{i+1}-s)^{1+\alpha}}{24(s-t_i)^2}\right], \tag{62}$$

then,

$$\sum_{i=0}^{m-1}\frac{h^3(s-t_i)^{1+\alpha}}{24(s-t_{i+1})^2} \leqslant \sum_{i=0}^{m-1}\frac{h^{4+\alpha}+h^3(s-t_{i+1})^{1+\alpha}}{(s-t_{i+1})^2} \leqslant h^{2+\alpha}\sum_{i=0}^{m-1}\frac{1+(m-i-1+(1+\tau)/2)^{1+\alpha}}{(m-i-1+(1+\tau)/2)^2} \leqslant \frac{Ch^2}{(1+\tau)^2}. \tag{63}$$

Similarly

$$\sum_{i=m+1}^{n-1}\frac{h^3(t_{i+1}-s)^{1+\alpha}}{24(s-t_i)^2} \leqslant \frac{Ch^2}{(1-\tau)^2}. \tag{64}$$

Putting (62), (63) and (64) together yields the last bound in (61), which complete the proof. $\square$

From now on, we give the proof of Theorem 2. By the Taylor expansion, there exists $\xi_i \in (t_i,t_{i+1})$, such that

$$\varphi(t)-\varphi(\hat{t}_i) = \varphi'(\xi_i)(t-\hat{t}_i), \quad t\in[t_i,t_{i+1}].$$

Then, by the means value theorem of integration, we have

$$\begin{aligned}\sum_{\substack{i=0\\i\neq m}}^{n-1}\int_{t_i}^{t_{i+1}}[\varphi(t)-\varphi(\hat{t}_i)]\cot\frac{t-s}{2}dt &= \sum_{\substack{i=0\\i\neq m}}^{n-1}\int_{t_i}^{t_{i+1}}\varphi'(\xi_i)(t-t_i)\cot\frac{t-s}{2}dt-\frac{h}{2}\sum_{\substack{i=0\\i\neq m}}^{n-1}\int_{t_i}^{t_{i+1}}\varphi'(\xi_i)\cot\frac{t-s}{2}dt\\
&= \sum_{\substack{i=0\\i\neq m}}^{n-1}\varphi'(\eta_i)\int_{t_i}^{t_{i+1}}(t-t_i)\cot\frac{t-s}{2}dt-\frac{h}{2}\sum_{\substack{i=0\\i\neq m}}^{n-1}\varphi'(\zeta_i)\int_{t_i}^{t_{i+1}}\cot\frac{t-s}{2}dt\\
&= \sum_{\substack{i=0\\i\neq m}}^{n-1}\varphi'(\eta_i)\int_{t_i}^{t_{i+1}}(t-\hat{t}_i)\cot\frac{t-s}{2}dt+\frac{h}{2}\sum_{\substack{i=0\\i\neq m}}^{n-1}[\varphi'(\eta_i)-\varphi'(\zeta_i)]\int_{t_i}^{t_{i+1}}\cot\frac{t-s}{2}dt\\
&= \sum_{\substack{i=0\\i\neq m}}^{n-1}[\varphi'(\eta_i)-\varphi'(s)]\mathcal{I}_{n,i}(s)+\varphi'(s)\sum_{\substack{i=0\\i\neq m}}^{n-1}\mathcal{I}_{n,i}(s)+h\sum_{\substack{i=0\\i\neq m}}^{n-1}[\varphi'(\eta_i)\\
&\quad-\varphi'(\zeta_i)]\left(\ln\left|\sin\frac{t_{i+1}-s}{2}\right|-\ln\left|\sin\frac{t_i-s}{2}\right|\right). \qquad (65)\end{aligned}$$

Let

$$\mathcal{G}_m(t)=\varphi(t)-\varphi(\hat{t}_m)-\varphi'(s)(t-\hat{t}_m),\quad t\in[t_m,t_{m+1}],$$

then

$$⨍_{t_m}^{t_{m+1}}[\varphi(t)-\varphi(\hat{t}_m)]\cot\frac{t-s}{2}dt=⨍_{t_m}^{t_{m+1}}\mathcal{G}_m(t)\cot\frac{t-s}{2}dt+\varphi'(s)\mathcal{I}_{n,m}(s). \qquad (66)$$

From (65), (66) and Lemma 10, we obtain

$$\mathcal{R}(s)=\mathcal{R}_1+\mathcal{R}_2+\mathcal{R}_3+\frac{h}{\pi}\varphi'(s)\ln\left(2\cos\frac{\tau\pi}{2}\right),$$

where

$$\begin{aligned}\mathcal{R}_1&=\frac{1}{2\pi}⨍_{t_m}^{t_{m+1}}\mathcal{G}_m(t)\cot\frac{t-s}{2}dt,\\
\mathcal{R}_2&=\frac{1}{2\pi}\sum_{\substack{i=0\\i\neq m}}^{n-1}[\varphi'(\eta_i)-\varphi'(s)]\mathcal{I}_{n,i}(s),\\
\mathcal{R}_3&=\frac{h}{2\pi}\sum_{\substack{i=0\\i\neq m}}^{n-1}[\varphi'(\eta_i)-\varphi'(\zeta_i)]\left(\ln\left|\sin\frac{t_{i+1}-s}{2}\right|-\ln\left|\sin\frac{t_i-s}{2}\right|\right).\end{aligned}$$

Now we estimate $\mathcal{R}_i$ term by term. Since $\varphi\in C^{2+\alpha}[0,2\pi](0<\alpha<1)$, we get

$$|\mathcal{G}_m^{(l)}(s)|\leqslant Ch^{2-l+\alpha},\quad l=0,1.$$

Then, by using the identity

$$⨍_a^b\psi(t)\cot\frac{t-s}{2}dt=2\psi(s)\left(\ln|\sin\frac{b-s}{2}|-\ln|\sin\frac{s-a}{2}|\right)+\int_a^b[\psi(t)-\psi(s)]\cot\frac{t-s}{2}dt,$$

we have

$$|\mathcal{R}_1|\leqslant\frac{1}{\pi}|\mathcal{G}_m(s)|\left|\ln\left|\sin\frac{t_{m+1}-s}{2}\right|-\ln\left|\sin\frac{s-t_m}{2}\right|\right|+\frac{1}{2\pi}\left|\int_a^b\mathcal{G}_m'(\delta_t)(t-s)\cot\frac{t-s}{2}dt\right|\leqslant C|\ln\gamma(h,s)|h^{2+\alpha},$$

where $\delta_t\in(t_m,t_{m+1})$. The second term $\mathcal{R}_2$ can be directly got by Lemma 10. Since $u\in C^{2+\alpha}[0,2\pi](0<\alpha<1)$, by direct calculation, we get

$$|\mathcal{R}_3|\leqslant C|\ln\gamma(h,s)|h^{2+\alpha}.$$

Putting above estimates together leads to Theorem 2.

6. Numerical experiments

In this section, we present some numerical examples to demonstrate the accuracy of collocation schemes for different equations (1). The notations of the schemes investigated are shown in Table 1. Moreover, we also give an example to verify the error estimates in Theorems 2 and 6.

Example 1. Consider the integral equation (1) with kernel $\kappa(t-s) = \frac{1}{2\pi}\cot\frac{t-s}{2}$ and the right term $g(s) = 2\cos 3s - \sin 2s$. The exact solution is $\varphi(t) = \cos 2t + 2\sin 3t$. We use collocation schemes Scheme I and Scheme II with different choice of collocation points to solve this integral equation and the corresponding discrete L^2-error is examined. Numerical results are presented in Table 2. The estimated orders of convergence are given in the last row, which are obtained by a least squares fit.

From the left half of Table 2, we see that the accuracy of Scheme I with $\tau = 2/3$ is $O(h^2)$, since the collocation points are chosen to be the superconvergence points in this case, which is in good agreement with our theoretical analysis. However, we can also see that the accuracy can only achieve $O(h)$ when the collocation points are not superconvergence points. Numerical results in the right half of Table 2 show that the accuracy of Scheme II with different τ are all $O(h^2)$.

Example 2. Next, we consider an example of solving integral equation (1) with kernel $\kappa(t-s) = \frac{1}{4\pi}\frac{1}{\sin^2\frac{t-s}{2}}$ and the right term $g(s) = -2(\cos 2s + \sin 2s)$. The exact solution of this problem is $\varphi(t) = \cos 2t + \sin 2t$.

We use collocation schemes Scheme III and Scheme IV to solve this hypersingular integral equation. Numerical results are presented in Table 3. From Table 3, we see that Scheme III is divergent and Scheme IV is only a first-order method in general, and the accuracy of both are $O(h^2)$ if their collocation points are chosen as the corresponding superconvergence points.

Example 3. At last, we give an example to verify the accuracy results in Theorem 2 and Theorem 6. Consider the Cauchy singular integral $\mathcal{H}_1\varphi$ defined in (16) with a smooth density function $\varphi(t) = 3\cos 3t + 2\sin 2t + 2$. The exact value of this Cauchy principle value integral is $2\cos 2s - 3\sin 3s$. We use composite midpoint rule (17) and trapezoidal rule (35) to evaluate $\mathcal{H}_1\varphi$. Numerical results are presented in the left of Tables 4 and Table 5 for a dynamic singular point $s = t_0 + (1+\tau)h/2$ and in the right of Tables for $s = t_{[n/4]} + (1+\tau)h/2$. In the first case singular point s approaches the startpoints as h goes to zero while in the second it is not close to the startpoints of the interval $[0, 2\pi]$. The estimated orders of convergence are given in the last row.

From Table 4, we see that if s is not a superconvergence point ($\tau \neq \pm 2/3$), the accuracy of midpoint rule is $O(h)$, while the accuracy can achieve $O(h^2)$ if s is located at the superconvergence points ($\tau = \pm 2/3$). These results agree quite well with the theoretical results in Theorems 2 and 3. Numerical results in Table 5 show that the accuracy of trapezoidal rule for evaluating $\mathcal{H}_1\varphi$ are $O(h^2)$ at all points.

Table 1
The notations for the collocation scheme (10) for solving equation (1) used in the numerical examples.

Notations	kernel $\kappa(t)$	Quadrature rule
Scheme I	$\frac{1}{2\pi}\cot\frac{t}{2}$	Midpoint rule (17)
Scheme II	$\frac{1}{2\pi}\cot\frac{t}{2}$	Trapezoidal rule (35)
Scheme III	$\frac{1}{4\pi}\frac{1}{\sin^2(t/2)}$	Midpoint rule (46)
Scheme IV	$\frac{1}{4\pi}\frac{1}{\sin^2(t/2)}$	Trapezoidal rule (52)

Table 2
L_2 Errors of collocation scheme for solving equation (1) with collocation points $\{s_i = t_i + (1+\tau)h/2\}_{i=0}^{n-1}$.

n	Scheme I			Scheme II		
	$\tau^\star = 2/3$	$\tau = 0$	$\tau = 1/2$	$\tau^\star = 0$	$\tau = 1/2$	$\tau = 2/3$
32	8.2560e−2	2.8038e+0	3.2246e−1	9.5700e−2	1.2133e−1	1.2998e−1
64	2.0705e−2	2.7402e−1	1.3978e−1	2.5019e−2	3.0046e−2	3.1500e−2
128	5.1913e−3	2.2865e−1	6.5110e−2	6.4031e−3	7.5024e−3	7.7833e−3
256	1.3002e−3	6.2839e−2	3.1430e−2	1.6203e−3	1.8762e−3	1.9362e−3
512	3.2538e−4	3.1105e−2	1.5443e−2	4.0759e−4	4.6921e−4	4.8297e−4
h^α	1.9967	1.5113	1.0921	1.9699	2.0030	2.0168

Table 3
L_2 Errors of collocation scheme for solving equation (1) with collocation points $\{s_i = t_i + (1+\tau)h/2\}_{i=0}^{n-1}$.

n	Scheme III			Scheme IV		
	$\tau^\star = 0$	$\tau = 1/2$	$\tau = 2/3$	$\tau^\star = 2/3$	$\tau = 0$	$\tau = 1/2$
32	1.6179e−2	1.5976e+0	2.0062e+0	3.7673e−2	2.7540e−1	1.4879e−1
64	4.0311e−3	1.6853e+0	2.0887e+0	9.4259e−3	1.2251e−1	6.3928e−2
128	1.0069e−3	1.7289e+0	2.1298e+0	2.3601e−3	5.7695e−2	2.9505e−2
256	2.5168e−4	1.7507e+0	2.1503e+0	5.9067e−4	2.8518e−2	1.4157e−2
512	6.2917e−5	1.7616e+0	2.1605e+0	1.4776e−4	1.3792e−2	6.9320e−3
h^α	2.0014	–	–	1.9985	1.0742	1.1023

Table 4
Errors of composite midpoint rule (17).

n	$s = t_0 + (1+\tau)h/2$			$s = t_{[n/4]} + (1+\tau)h/2$		
	$\tau = -2/3$	$\tau = 0$	$\tau = 2/3$	$\tau = -2/3$	$\tau = 0$	$\tau = 2/3$
64	1.0831e-2	5.9221e-2	1.4152e-2	6.8847e-3	1.1401e-1	9.7472e-3
128	2.6279e-3	3.6699e-2	3.8660e-3	1.8156e-3	5.5826e-2	2.1930e-3
256	6.4641e-4	2.0036e-2	1.0033e-3	4.6547e-4	2.7524e-2	5.1379e-4
512	1.6025e-4	1.0428e-2	2.5515e-4	1.1780e-4	1.3654e-2	1.2391e-4
1024	3.9891e-5	5.3151e-3	6.4308e-5	2.9628e-5	6.7984e-3	3.0396e-5
h^α	2.0205	0.8771	1.9485	1.9667	1.0167	2.0795

Table 5
Errors of composite trapezoidal rule (35)

n	$s = t_0 + (1+\tau)h/2$			$s = t_{[n/4]} + (1+\tau)h/2$		
	$\tau = -2/3$	$\tau = 0$	$\tau = 2/3$	$\tau = -2/3$	$\tau = 0$	$\tau = 2/3$
64	1.8903e−2	3.1578e−3	1.2555e−2	1.6106e−2	1.4122e−2	1.2425e−2
128	4.8427e−3	1.1882e−3	2.4580e−3	3.9258e−3	3.6877e−3	3.4972e−3
256	1.2245e−3	3.4877e−4	5.2560e−4	9.6793e−4	9.3882e−4	9.1641e−4
512	3.0782e−4	9.3763e−5	1.2009e−4	2.4023e−4	2.3664e−4	2.3392e−4
1024	7.7162e−5	2.4268e−5	2.8599e−5	5.9836e−5	5.9390e−5	5.9056e−5
h^α	1.9849	1.7711	2.1911	2.0175	1.9749	1.9336

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