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Nodal-Type Newton-Cotes Rules for Fractional Hypersingular Integrals

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Abstract. Nodal-type Newton-Cotes rules for fractional hypersingular integrals based on the piecewise k-th order Newton interpolations are proposed. A general error estimate is first derived on quasi-uniform meshes and then we show that the even-order rules exhibit the superconvergence phenomenon — i.e. if the singular point is far away from the endpoints then the accuracy of the method is one order higher than the general estimate. Numerical experiments confirm the theoretical results.

AMS subject classifications: 65D30, 65D32, 45E99

Key words: Hypersingular integrals, fractional order, nodal-type Newton-Cotes rules, superconvergence.

1. Introduction

Considering the integral

$$\mathscr{I}u(x) = \,=\!\!\!\!\!\!\int_a^b \frac{u(y)}{|y-x|^{1+2s}}\,dy, \quad s \in [0,1), \quad x \in (a,b), \tag{1.1}$$

we note that it does not exist in usual sense and should be specifically defined. These types of integrals are often referred to as Hadamard finite-part integrals or hypersingular integrals. There are various definitions and we first consider the case where the singular point is located at an interval end — cf. [22]. In this case the integral can be defined as

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$$
\begin{aligned}
&⨍_a^x \frac{u(y)}{(x-y)^{1+2s}}\,dy := \lim_{\epsilon\to 0}\left(\int_a^{x-\epsilon} \frac{u(y)}{(x-y)^{1+2s}}\,dy + r_-(\epsilon)\right),\\
&⨍_x^b \frac{u(y)}{(y-x)^{1+2s}}\,dy := \lim_{\epsilon\to 0}\left(\int_{x+\epsilon}^{b} \frac{u(y)}{(y-x)^{1+2s}}\,dy + r_+(\epsilon)\right),
\end{aligned}
\tag{1.2}
$$

where

$$
r_-(\epsilon) = \begin{cases}
u(x_-)\ln\epsilon, & s=0,\\
u(_-)\dfrac{\epsilon^{-2s}}{-2s}, & s\in(0,1/2),\\
-u(x_-)\epsilon^{-1} - u'(x_-)\ln\epsilon, & s=1/2,\\
u(x_-)\dfrac{\epsilon^{-2s}}{-2s} - u'(x_-)\dfrac{\epsilon^{1-2s}}{1-2s}, & s\in(1/2,1),
\end{cases}
$$

$$
r_+(\epsilon) = \begin{cases}
u(x_+)\ln\epsilon, & s=0,\\
u(x_+)\dfrac{\epsilon^{-2s}}{-2s}, & s\in(0,1/2),\\
-u(x_+)\epsilon^{-1} + u'(x_+)\ln\epsilon, & s=1/2,\\
u(x_+)\dfrac{\epsilon^{-2s}}{-2s} + u'(x_+)\dfrac{\epsilon^{1-2s}}{1-2s}, & s\in(1/2,1),
\end{cases}
$$

and $u(x_-)$ and $u(x_+)$ are, respectively, the left and right limits of u at x. If $x\in(a,b)$, then we define the corresponding integral as

$$
\mathscr{I}u(x) := \lim_{\epsilon\to 0}\left[\left(\int_a^{x-\epsilon} + \int_{x+\epsilon}^{b}\right)\frac{u(y)}{|y-x|^{1+2s}}\,dy + r(\epsilon)\right], \quad x\in(a,b), \tag{1.3}
$$

where

$$
r(\epsilon) = r_-(\epsilon) + r_+(\epsilon).
$$

A function $u(y)$ is said to be Hadamard finite-part integrable with respect to the weight $|y-x|^{-1-2s}$ if the limit in the right-hand side of (1.3) exists. It is worth noting that if $u(y)$ has a strong regularity, then $r(\epsilon)$ can be represented as

$$
r(\epsilon) = u(x)\begin{cases}
2\ln\epsilon, & s=0,\\
-\dfrac{\epsilon^{-2s}}{s}, & s\in(0,1).
\end{cases}
$$

The approximation of hypersingular integrals plays an important role in numerical methods for various integral equations arising in acoustics [27], electromagnetics [20,26], heat conduction [18]. Besides, equations with hypersingular integrals are also used in stress calculation [3,9], fracture mechanics [1,2,4,8] and wave scattering [2,11,12]. A special attention has been paid to quadrature formulas for hypersingular integrals, including Gaussian quadratures [10,15,17,21], composite Newton-Cotes rules [6,16,23,24] and transformation methods [5,7]. The Gaussian quadratures demonstrate a very good numerical performance for smooth density functions $u(y)$ and are often used for the construction of stiffness matrices in finite element methods. Nevertheless, in practical computations the composite Newton-Cotes rules are often preferable due to the easiness of implementation and high flexibility during the mesh construction. Thus they are employed in collocation method for Hadamard finite-part integral equations and nonlocal diffusion problems [19,28] and are used to discretise boundary hypersingular integral equations in the electromagnetic cavity problems [25].

This work follows the approach of [13] and [28], where nodal-type Newton-Cotes rules are applied to hypersingular integrals of integer and fractional orders, respectively. Here we consider the application of nodal-type Newton-Cotes rules based on piecewise Lagrange-Newton interpolations of (1.1) in the case $s \in [0,1)$ and establish error estimates for quasi-uniform meshes. In addition, we prove that the even-order rules exhibit the so-called superconvergence phenomenon — i.e. if the singular point is far away from the endpoints, then the accuracy of the method is one order higher than general estimates.

The rest of the paper is organised as follows. In Section 2, the fractional hypersingular integral (1.1) is approximated by nodal-type Newton-Cotes rules. In Section 3, we establish a general estimate of the method in the case of quasi-uniform meshes and show its superconvergence on uniform meshes for the quadratures of even order. The numerical examples presented in Section 4 confirm theoretical results and demonstrate the efficiency and accuracy of the method.

2. Newton-Cotes Formulas for Hypersingular Integrals

Let $a = x_0 < x_1 < \cdots < x_n < x_{n+1} = b$ be the partition of the interval $[a,b]$ with the grid size $h_j = x_{j+1} - x_j,\ j = 0,1,\cdots,n$ and let $h = \max_{0\le j\le n} h_j$. To define the piecewise Newton interpolation of order k, on each subinterval $[x_j, x_{j+1}]$ we introduce another partition $x_j = x_j^0 < x_j^1 < \cdots < x_j^k = x_{j+1}$. In what follows, we always assume that $x = x_i$ for an $1 \le i \le n$.

Introducing elementary symmetric polynomials

$$\sigma_j^m = \sigma_j^m(r_0, r_1, \cdots, r_{m-1}) = \begin{cases} 1, & j = 0, \\ \displaystyle\sum_{0\le i_0\le\cdots\le i_{j-1}\le m-1} \prod_{j'=0}^{j-1} r_{i_{j'}}, & j = 1,\cdots,m, \end{cases} \tag{2.1}$$

with respect to $r_0, \cdots, r_{m-1}$, we can rewrite the k-th degree Newton interpolation

$$\pi_h^{(k)} u(y) = u(x_j^0) + \sum_{\ell=1}^{k} u[x_j^0, \cdots, x_j^\ell] \prod_{m=0}^{\ell-1} (y - x_j^m), \quad j = 0,1,\cdots,n$$

of the function $u(y)$ on the subinterval $[x_j, x_{j+1}]$ as

$$\pi_h^{(k)} u(y) = \sum_{\ell=0}^{k} \delta_j^{(k,\ell)} u(x_i)(y-x_i)^\ell, \quad j=0,1,\cdots,n, \tag{2.2}$$

where

$$\delta_j^{(k,\ell)} u(x_i) = \begin{cases} u(x_j^0), & \ell = 0, \\ \sum_{m=\ell}^{k} u[x_j^0, x_j^1, \cdots, x_j^m] \sigma_{m-\ell}^m (x_i - x_j^0, \cdots, x_i - x_j^{m-1}), & \ell = 1, \cdots, k. \end{cases} \tag{2.3}$$

Substituting (2.2) into (1.1) yields the k-th order nodal-type Newton-Cotes formula

$$\mathcal{Q}_h^{(k)} u(x_i) = \left(\sum_{\substack{j=0,\\ j\neq i-1,i}}^{n} \int_{x_j}^{x_{j+1}} + \fint_{x_{i-1}}^{x_{i+1}} \right) \frac{\pi_h^{(k)} u(y)}{|y-x_i|^{1+2s}}\, dy = \sum_{j=0}^{n} \sum_{\ell=0}^{k} \delta_j^{(k,\ell)} u(x_i) \alpha_{ij}^{(\ell)}, \tag{2.4}$$

where

$$\alpha_{ij}^{(\ell)} = \begin{cases} \displaystyle\fint_{x_j}^{x_{j+1}} \frac{(y-x_i)^\ell}{|y-x_i|^{1+2s}}\, dy, & \text{if} \quad j \in \{i-1, i\}, \quad 0 \le \ell \le 2s, \\ \displaystyle\int_{x_j}^{x_{j+1}} \frac{(y-x_i)^\ell}{|y-x_i|^{1+2s}}\, dy, & \text{otherwise}. \end{cases} \tag{2.5}$$

The terms $\alpha_{ij}^{(\ell)}$ can be written explicitly. Thus straightforward calculation show that if $s \in (0,1/2) \cup (1/2,1)$, then

$$\alpha_{ij}^{(\ell)} = \frac{1}{\ell - 2s} \begin{cases} (x_{j+1} - x_i)^{\ell-2s} - (x_j - x_i)^{\ell-2s}, & j \ge i+1, \\ (x_{i+1} - x_i)^{\ell-2s}, & j = i, \\ (-1)^\ell (x_i - x_{i-1})^{\ell-2s}, & j = i-1, \\ (-1)^{\ell+1} \left[(x_i - x_{j+1})^{\ell-2s} - (x_i - x_j)^{\ell-2s} \right], & j < i-1. \end{cases} \tag{2.6}$$

3. Errors of Newton-Cotes Formulas

In this section, we study the errors of the nodal-type Newton-Cotes formulas (2.4). If u is a sufficiently smooth function defined on $[r_0, r_k]$, and $r_0 < r_1 < \cdots < r_k$ is a partition of the subinterval $[r_0, r_k]$, then for $1 \le \ell \le k$, we define the terms $\Delta^{(k,\ell)} u(r_0)$ and $\Delta^{(k,\ell)} u(r_k)$ by

$$\begin{aligned} \Delta^{(k,\ell)} u(r_0) &:= \sum_{m=\ell}^{k} u[r_0, \cdots, r_m] \sigma_{m-\ell}^{m-1} (r_0 - r_1, \cdots, r_0 - r_{m-1}), \\ \Delta^{(k,\ell)} u(r_k) &:= \sum_{m=\ell}^{k} u[r_{k-m}, \cdots, r_k] \sigma_{m-\ell}^{m-1} (r_k - r_{k-m+1}, \cdots, r_k - r_{k-1}). \end{aligned} \tag{3.1}$$

Theorem 3.1. *If $u \in C^{k+1}[r_0, r_k]$ with $k \geq 1$ and $\{r_j\}$ is a quasi-uniform mesh, then for $\Delta^{(k,\ell)}u(r_0)$ and $\Delta^{(k,\ell)}u(r_k)$ one has*

$$\frac{u^{(\ell)}(r_0)}{\ell!} = \Delta^{(k,\ell)}u(r_0) + \mathscr{R}_1, \quad \frac{u^{(\ell)}(r_k)}{\ell!} = \Delta^{(k,\ell)}u(r_k) + \mathscr{R}_2, \tag{3.2}$$

where

$$\mathscr{R}_1 = \begin{cases} -\dfrac{u''(\xi_0)}{2}(r_1 - r_0), \quad k = 1, \\ \dfrac{(-1)^k}{(k+1)!}\sum_{j=1}^{k}\dfrac{(r_0 - r_j)^{k+1}}{\omega_k'(r_j)}\sigma_{k-\ell}^{k-1}(\underbrace{r_0 - r_1, \cdots, r_0 - r_k}_{neglect\, r_0 - r_j})u^{(k+1)}(\xi_j), \\ \qquad k > 1 \text{ and } \ell = 1, 2, \cdots, k, \end{cases} \tag{3.3}$$

$$\mathscr{R}_2 = \begin{cases} \dfrac{u''(\zeta_0)}{2}(r_1 - r_0), \quad k = 1, \\ \dfrac{(-1)^k}{(k+1)!}\sum_{j=0}^{k-1}\dfrac{(r_k - r_j)^{k+1}}{\omega_k'(r_j)}\sigma_{k-\ell}^{k-1}(\underbrace{r_k - r_0, \cdots, r_k - r_{k-1}}_{neglect\ r_k - r_j})u^{(k+1)}(\zeta_j), \\ \qquad k > 1 \text{ and } \ell = 1, 2, \cdots, k, \end{cases} \tag{3.4}$$

$$\omega_k(r) = \prod_{j=0}^{k}(r - r_j), \tag{3.5}$$

and ξ_j depends on r_0 and r_j, ζ_j depends on r_j and r_k.

Proof. The proof of this theorem is given in Appendix. ☐

Replacing $\{r_\ell\}_{\ell=0}^k$ in (3.1) by $\{x_i^\ell\}_{\ell=0}^k$ and comparing the result with (2.3), we observe that $\delta_i^{(k,\ell)}u(x_i) = \Delta^{(k,\ell)}u(r_0)$. On the other hand, using an alternative representation

$$\pi_h^{(k)}u(y) = u(x_{i-1}^k) + \sum_{\ell=1}^{k}u[x_{i-1}^{k-\ell}, \cdots, x_{i-1}^k]\prod_{m=0}^{\ell-1}(y - x_{i-1}^{k-m}),$$

of the k-th degree Newton interpolation of $u(y)$ on $[x_{i-1}, x_i]$, we rewrite $\delta_{i-1}^{(k,\ell)}u(x_i)$ in (2.3) as

$$\delta_{i-1}^{(k,\ell)}u(x_i) = \begin{cases} u(x_i), \quad \ell = 0, \\ \sum_{m=\ell}^{k} u[x_{i-1}^{k-m}, \cdots, x_{i-1}^k]\sigma_{m-\ell}^{m-1}(x_{i-1}^k - x_{i-1}^{k-m+1}, \cdots, \quad x_{i-1}^k - x_{i-1}^{k-1}), \\ \qquad \ell = 1, \cdots, k. \end{cases}$$

Replacing $\{r_\ell\}_{\ell=0}^k$ in (3.1) by $\{x_{i-1}^\ell\}_{\ell=0}^k$ and comparing the result with the equation above, one obtains $\delta_{i-1}^{(k,\ell)}u(x_i) = \Delta^{(k,\ell)}u(r_k)$.

This leads to the following corollaries.

Corollary 3.1. *If $u \in C^{k+1}[a,b]$ and $\{x_j\}$, $\{x_j^\ell\}$ are quasi-uniform meshes, then for $\ell = 1,\cdots,k$ the relations*

$$\frac{u^{(\ell)}(x_i)}{\ell!} = \delta_i^{(k,\ell)} u(x_i) + \mathscr{O}(h^{k+1-\ell}), \quad \frac{u^{(\ell)}(x_i)}{\ell!} = \delta_{i-1}^{(k,\ell)} u(x_i) + \mathscr{O}(h^{k+1-\ell})$$

hold.

Corollary 3.2. *Let k be an even non-negative integer. If $u \in C^{k+2}[a,b]$ and $\{x_j\}$, $\{x_j^\ell\}$ are uniform meshes, then the relations*

$$\begin{aligned}
&\frac{u^{(2\ell)}(x_i)}{(2\ell)!} = \frac{\delta_{i-1}^{(k,2\ell)} u(x_i) + \delta_i^{(k,2\ell)} u(x_i)}{2} + \mathscr{O}(h^{k+2-2\ell}), && \ell = 1,\cdots,k/2, \\
&\delta_{i-1}^{(k,2\ell-1)} u(x_i) - \delta_i^{(k,2\ell-1)} u(x_i) = \mathscr{O}(h^{k+3-2\ell}), && \ell = 1,\cdots,k/2
\end{aligned}$$

hold.

3.1. Error estimates for quasi-uniform meshes

Now we evaluate the approximations (2.4) in the case of quasi-uniform meshes.

Theorem 3.2. *If $u \in C^{k+1}[a,b]$, $\{x_j\}$, $\{x_j^\ell\}$ are quasi-uniform meshes and $\mathscr{Q}_h^{(k)} u(x_i)$ is defined by (2.4), then*

$$|\mathscr{I} u(x_i) - \mathscr{Q}_h^{(k)} u(x_i)| \le C \|u^{(k+1)}\|_\infty \begin{cases} h^{k+1}|\ln h|, & s = 0, \\ h^k |\ln h|, & s = 1/2, \\ h^{k+1-2s}, & \text{otherwise} \end{cases}$$

for any $i = 1,2,\cdots,n$.

Proof. Here we only consider the case $s \in (0,1/2) \cup (1/2,1)$. For other values of s the proof is analogous. Set $e(y) := u(y) - \pi_h^{(k)} u(y)$ and represent the difference $\mathscr{I} u(x_i) - \mathscr{Q}_h^{(k)} u(x_i)$ in the form

$$\mathscr{I} u(x_i) - \mathscr{Q}_h^{(k)} u(x_i) = \mathscr{E}_1 + \mathscr{E}_2 + \mathscr{E}_3 + \mathscr{E}_4, \tag{3.6}$$

where

$$\begin{aligned}
\mathscr{E}_1 &= \fint_{x_i}^{x_{i+1}} \frac{e(y)}{(y - x_i)^{1+2s}} \, dy, \quad \mathscr{E}_2 = \fint_{x_{i-1}}^{x_i} \frac{e(y)}{(x_i - y)^{1+2s}} \, dy, \\
\mathscr{E}_3 &= \int_{x_{i+1}}^{b} \frac{e(y)}{(y - x_i)^{1+2s}} \, dy, \quad \mathscr{E}_4 = \int_a^{x_{i-1}} \frac{e(y)}{(x_i - y)^{1+2s}} \, dy.
\end{aligned}$$

Let us estimate each of integrals $\mathscr{E}_m, m=1,2,3,4$. We start with the terms $\mathscr{E}_1$ and $\mathscr{E}_2$. Using (2.2) and the identity $\delta_i^{(k,0)}u(x_i)=u(x_i)$, we write $\mathscr{E}_1$ as

$$\mathscr{E}_1=\mathscr{E}_1^*+\mathscr{E}_1^{**}, \tag{3.7}$$

where

$$\mathscr{E}_1^*=\int_{x_i}^{x_{i+1}}\left[u(y)-\sum_{\ell=0}^{k}\frac{u^{(\ell)}(x_i)(y-x_i)^\ell}{\ell!}\right]\frac{1}{(y-x_i)^{1+2s}}\,dy,$$
$$\mathscr{E}_1^{**}=\sum_{\ell=1}^{k}\left[\frac{u^{(\ell)}(x_i)}{\ell!}-\delta_i^{(k,\ell)}u(x_i)\right]\alpha_{ii}^{(\ell)}.$$

Since $u\in C^{k+1}[a,b]$, it follows that

$$\left|\mathscr{E}_1^*\right|=\left|\int_{x_i}^{x_{i+1}}\frac{u^{(k+1)}(\xi)}{(k+1)!}(y-x_i)^{k-2s}\,dy\right|\le C\|u^{(k+1)}\|_\infty h^{k+1-2s}. \tag{3.8}$$

Moreover, (2.6) and Corollary 3.1 yield

$$\left|\mathscr{E}_1^{**}\right|\le\sum_{\ell=1}^{k}\left|\frac{u^{(\ell)}(x_i)}{\ell!}-\delta_i^{(k,\ell)}u(x_i)\right|\left|\alpha_{ii}^{(\ell)}\right|\le C\|u^{(k+1)}\|_\infty h^{k+1-2s}. \tag{3.9}$$

Combining (3.7), (3.8) and (3.9), we obtain

$$|\mathscr{E}_1|\le C\|u^{(k+1)}\|_\infty h^{k+1-2s}. \tag{3.10}$$

The term $\mathscr{E}_2$ can be estimated analogously, so that

$$|\mathscr{E}_2|\le C\|u^{(k+1)}\|_\infty h^{k+1-2s}. \tag{3.11}$$

We next estimate $\mathscr{E}_3$ and $\mathscr{E}_4$. The representation

$$e_j(y)=\frac{u^{(k+1)}(\xi)}{(k+1)!}\prod_{\ell=0}^{k}(y-x_j^\ell),\quad y,\xi\in[x_j,x_{j+1}], \tag{3.12}$$

for $j=i+1,\cdots,n$ implies that

$$\begin{aligned}|\mathscr{E}_3|&\le\frac{C\|u^{(k+1)}\|_\infty h^{k+1}}{(k+1)!}\int_{x_{i+1}}^{b}(y-x_i)^{-1-2s}\,dy\\&=\frac{C\|u^{(k+1)}\|_\infty h^{k+1}}{(k+1)!}\frac{h^{-2s}-(b-x_i)^{-2s}}{2s}\le C\|u^{(k+1)}\|_\infty h^{k+1-2s}.\end{aligned} \tag{3.13}$$

Similarly, we obtain

$$|\mathscr{E}_4|\le C\|u^{(k+1)}\|_\infty h^{k+1-2s}. \tag{3.14}$$

The combination of (3.6), (3.10), (3.11), (3.13) and (3.14) completes the proof. $\square$

3.2. Superconvergence of even-order rules

Here we show that if $\{x_j\}$, $\{x_j^\ell\}$ are uniform meshes, then the even-order rules exhibit superconvergence phenomenon. We start with auxiliary results.

Lemma 3.1. *If $\{x_j\}$, $\{x_j^\ell\}$ are uniform meshes and $\alpha_{ij}^{(\ell)}$ are defined by* (2.5)*, then*

$$\alpha_{i,i-1}^{(\ell)} = (-1)^\ell \alpha_{ii}^{(\ell)} \tag{3.15}$$

and

$$|\alpha_{ii}^{(\ell)}| = \begin{cases} \mathcal{O}(|\ln h|), & \ell = 2s, \\ \mathcal{O}(h^{\ell-2s}), & \text{otherwise}. \end{cases}$$

Proof. It follows immediately from the definition of $\alpha_{ij}^{(\ell)}$. □

For $1 \le j \le n$, we set

$$\Phi_{ij}^{(k)} := \begin{cases} -\displaystyle\int_0^1 \frac{\prod_{\ell=0}^k (t-\ell/k)}{(t+i-j)^{1+2s}}\,dt, & 1 \le j < i, \\ 0, & j = i, \\ \displaystyle\int_0^1 \frac{\prod_{\ell=0}^k (t-\ell/k)}{(t+j-i)^{1+2s}}\,dt, & i < j \le n. \end{cases} \tag{3.16}$$

Using a variable transformation, we obtain the following lemma.

Lemma 3.2. *If $\Phi_{ij}^{(k)}$ is defined by* (3.16)*, then*

$$h^{k+1-2s}\Phi_{ij}^{(k)} = \begin{cases} \displaystyle\int_{x_{j-1}}^{x_j} \frac{\prod_{\ell=0}^k (y-x_{j-1}^\ell)}{(x_i-y)^{1+2s}}\,dy, & 1 \le j < i, \\ \displaystyle\int_{x_j}^{x_{j+1}} \frac{\prod_{\ell=0}^k (y-x_j^\ell)}{(y-x_i)^{1+2s}}\,dy, & i < j \le n. \end{cases}$$

Now we can evaluate the accuracy of the approximation (2.4).

Theorem 3.3. *Let k be an even non-negative integer. If $u \in C^{k+2}[a,b]$, $\{x_j\}$ and $\{x_j^\ell\}$ are uniform meshes and $\mathcal{Q}_h^{(k)}u(x_i)$ is defined by* (2.4)*, then*

$$\mathcal{I}u(x_i) - \mathcal{Q}_h^{(k)}u(x_i) = h^{k+1-2s}\sum_{j=1}^n \frac{u^{(k+1)}(x_j)}{(k+1)!}\Phi_{ij}^{(k)} + \mathcal{R}_i(u), \tag{3.17}$$

where

$$|\mathcal{R}_i(u)| \le \begin{cases} Ch^{k+2}|\ln h|, & s = 0, \\ Ch^{k+1}|\ln h|, & s = 1/2, \\ Ch^{k+2-2s}, & \text{otherwise}. \end{cases}$$

Proof. We again use the representation (3.6) and only consider the case $s \in (0,1/2) \cup (1/2,1)$. Starting with the terms $\mathscr{E}_1 + \mathscr{E}_2$ and using the Taylor expansion along with (3.15), we write

$$\begin{aligned}
\fint_{x_{i-1}}^{x_{i+1}} \frac{u(y)}{|y-x_i|^{1+2s}}\,dy &= 2u(x_i)\alpha_{ii}^{(0)} + 2\sum_{\ell=1}^{k/2} \frac{u^{(2\ell)}(x_i)}{(2\ell)!}\alpha_{ii}^{(2\ell)} \\
&\quad + \int_{x_{i-1}}^{x_{i+1}} \left(u(y) - \sum_{\ell=0}^{k+1} \frac{u^{(\ell)}(x_i)(y-x_i)^\ell}{\ell!} \right) \frac{1}{|y-x_i|^{1+2s}}\,dy. \qquad (3.18)
\end{aligned}$$

On the other hand, it follows from the Eq. (2.2) that

$$\begin{aligned}
\fint_{x_{i-1}}^{x_{i+1}} \frac{\pi_h^{(k)} u(y)}{|y-x_i|^{1+2s}}\,dy &= 2u(x_i)\alpha_{ii}^{(0)} + \sum_{\ell=1}^{k/2} \Big[\delta_{i-1}^{(k,2\ell)} u(x_i) + \delta_i^{(k,2\ell)} u(x_i) \Big] \alpha_{ii}^{(2\ell)} \\
&\quad - \sum_{\ell=1}^{k/2} \Big[\delta_{i-1}^{(k,2\ell-1)} u(x_i) - \delta_i^{(k,2\ell-1)} u(x_i) \Big] \alpha_{ii}^{(2\ell-1)}, \qquad (3.19)
\end{aligned}$$

and subtracting (3.19) from (3.18) yields

$$\begin{aligned}
\mathscr{E}_1 + \mathscr{E}_2 &= 2\sum_{\ell=1}^{k/2} \left[\frac{u^{(2\ell)}(x_i)}{(2\ell)!} - \frac{\delta_{i-1}^{(k,2\ell)} u(x_i) + \delta_i^{(k,2\ell)} u(x_i)}{2} \right] \alpha_{ii}^{(2\ell)} \\
&\quad + \sum_{\ell=1}^{k/2} \Big[\delta_{i-1}^{(k,2\ell-1)} u(x_i) - \delta_i^{(k,2\ell-1)} u(x_i) \Big] \alpha_{ii}^{(2\ell-1)} \\
&\quad + \int_{x_{i-1}}^{x_{i+1}} \left(u(y) - \sum_{\ell=0}^{k+1} \frac{u^{(\ell)}(x_i)(y-x_i)^\ell}{\ell!} \right) \frac{1}{|y-x_i|^{1+2s}}\,dy. \qquad (3.20)
\end{aligned}$$

It follows from Corollary 3.2 and Lemma 3.1, that all terms in the first and second sums in the right-hand side of (3.20) behave as $\mathscr{O}(h^{k+2-2s})$. Moreover, estimating the Taylor formula remainder, one can show that the integral in (3.20) is bounded by $\mathscr{O}(h^{k+2-2s})$. Hence

$$|\mathscr{E}_1 + \mathscr{E}_2| \le Ch^{k+2-2s}. \qquad (3.21)$$

Next, we estimate $\mathscr{E}_3$ and $\mathscr{E}_4$. The Eq. (3.12) shows that for the interval $[x_j, x_{j+1}], j = i+1, \cdots, n$ the error of Newton interpolation can be written as

$$e_j(y) = \frac{u^{(k+1)}(\xi_j) - u^{(k+1)}(x_j)}{(k+1)!} \prod_{\ell=0}^{k} (y - x_j^\ell) + \frac{u^{(k+1)}(x_j)}{(k+1)!} \prod_{\ell=0}^{k} (y - x_j^\ell), \quad \xi_j \in [x_j, x_{j+1}].$$

Taking into account that $u \in C^{k+2}[a,b]$, we obtain

$$\left| \sum_{j=i+1}^{n} \int_{x_j}^{x_{j+1}} \frac{u^{(k+1)}(\xi_j) - u^{(k+1)}(x_j)}{(k+1)!} \frac{\prod_{\ell=0}^{k} (y - x_j^\ell)}{(y-x_i)^{1+2s}}\,dy \right|$$

$$
\begin{aligned}
&\le \frac{C\|u^{(k+2)}\|_\infty h^{k+2}}{(k+1)!}\sum_{j=i+1}^{n}\int_{x_j}^{x_{j+1}}\frac{1}{(y-x_i)^{1+2s}}dy\\
&= \frac{Ch^{k+2}\|u^{(k+2)}\|_\infty}{(k+1)!}\frac{h^{-2s}-(b-x_i)^{-2s}}{2s}\le Ch^{k+2-2s}.
\end{aligned}
$$

Therefore, Lemma 3.2 implies that

$$
\mathscr{E}_3 = h^{k+1-2s}\sum_{j=i+1}^{n}\frac{u^{(k+1)}(x_j)}{(k+1)!}\Phi_{ij}^{(k)} + \mathscr{O}(h^{k+2-2s}). \tag{3.22}
$$

Similarly, we have

$$
\mathscr{E}_4 = h^{k+1-2s}\sum_{j=1}^{i-1}\frac{u^{(k+1)}(x_j)}{(k+1)!}\Phi_{ij}^{(k)} + \mathscr{O}(h^{k+2-2s}), \tag{3.23}
$$

and the representation (3.17) follows from the relations (3.22), (3.23) and (3.21). $\square$

Actually, a more detailed analysis allows to obtain better convergence estimates.

Theorem 3.4. *Under the conditions of Theorem* 3.3*, one has*

$$
|\mathscr{I}u(x_i)-\mathscr{Q}_h^{(k)}u(x_i)|\le Ch^{k+2}\eta_i^{-1-2s}+\begin{cases} Ch^{k+2}|\ln h|, & s=0,\\ Ch^{k+1}|\ln h|, & s=1/2,\\ Ch^{k+2-2s}, & \textit{otherwise}\end{cases}
$$

for $i=1,2,\cdots,n$*, where*

$$
\eta_i = \min(x_i-a, b-x_i).
$$

Proof. We want to derive better estimates for the sum in the right-hand side of (3.17). Recall that we still consider the case $s\in(0,1/2)\cup(1/2,1)$. Introducing the sums

$$
\begin{aligned}
S_j &:= \sum_{l=1}^{j}\Phi_{il}^{(k)}, \quad j=1,\cdots,i-1,\\
\bar{S}_j &= \sum_{l=j}^{n}\Phi_{il}^{(k)}, \quad j=i+1,\cdots,n,
\end{aligned}
$$

we write

$$
\begin{aligned}
\sum_{j=1}^{i-1}\frac{u^{(k+1)}(x_j)}{(k+1)!}\Phi_{ij}^{(k)} &= \frac{u^{(k+1)}(x_i)}{(k+1)!}S_{i-1}-\sum_{j=1}^{i-1}\frac{u^{(k+1)}(x_{j+1})-u^{(k+1)}(x_j)}{(k+1)!}S_j\\
&= \frac{u^{(k+1)}(x_i)}{(k+1)!}S_{i-1}-h\sum_{j=1}^{i-1}\frac{u^{(k+2)}(\xi_j)}{(k+1)!}S_j,
\end{aligned}
$$

$$\sum_{j=i+1}^{n} \frac{u^{(k+1)}(x_j)}{(k+1)!}\Phi_{ij}^{(k)} = \frac{u^{(k+1)}(x_i)}{(k+1)!}\bar{S}_{i+1} + h\sum_{j=i+1}^{n} \frac{u^{(k+2)}(\xi_j)}{(k+1)!}\bar{S}_j,$$

and since $\Phi_{ii}^{(k)} = 0$, it follows that

$$\sum_{j=1}^{n} \frac{u^{(k+1)}(x_j)}{(k+1)!}\Phi_{ij}^{(k)} = \frac{u^{(k+1)}(x_i)}{(k+1)!}\sum_{j=1}^{n}\Phi_{ij}^{(k)} - h\sum_{j=1}^{i-1}\frac{u^{(k+2)}(\xi_j)}{(k+1)!}S_j + h\sum_{j=i+1}^{n}\frac{u^{(k+2)}(\xi_j)}{(k+1)!}\bar{S}_j. \quad (3.24)$$

If k is an even natural number, then

$$\int_0^1 \prod_{\ell=0}^{k}(t-\ell/k)\, dt = 0,$$

and integration by parts yields

$$\int_0^1 \frac{\prod_{\ell=0}^{k}(t-\ell/k)}{(i-j+1-t)^{1+2s}}\, dt = -(1+2s)\int_0^1 \frac{\int_0^t \prod_{\ell=0}^{k}(t-\ell/k)\, d\tau}{(i-j+1-t)^{2+2s}}\, dt.$$

Hence,

$$|S_j| \le C\sum_{l=1}^{j}\frac{1}{(i-l)^{2+2s}} \le C\frac{1}{(i-j)^{1+2s}},$$

so that the sum $\sum_{j=1}^{i-1}|S_j|$ is uniformly bounded. Similarly, one shows that the sum $\sum_{j=i+1}^{n}|\bar{S}_j|$ is also uniformly bounded. Therefore, it follows from (3.24) that

$$\sum_{j=1}^{n}\frac{u^{(k+1)}(x_i)}{(k+1)!}\Phi_{ij}^{(k)} = \frac{u^{(k+1)}(x_i)}{(k+1)!}\sum_{j=1}^{n}\Phi_{ij}^{(k)} + \mathcal{O}(h).$$

Without loss of generality, we assume that $i \ge (n+1)/2$. It follows from (3.16) that $\Phi_{i,i-m}^{(k)} = -\Phi_{i,i+m}^{(k)}$, hence

$$\left|\sum_{j=1}^{n}\Phi_{ij}^{(k)}\right| = \left|\sum_{j=1}^{2i-n-1}\Phi_{ij}^{(k)}\right| = |S_{2i-n-1}| \le C(n+1-i)^{-1-2s} \le Ch^{1+2s}\eta_i^{-1-2s}.$$

Consequently,

$$\sum_{j=1}^{n}\frac{u^{(k+1)}(x_i)}{(k+1)!}\Phi_{ij}^{(k)} = \mathcal{O}(h^{1+2s}\eta_i^{-1-2s}) + \mathcal{O}(h).$$

The above estimates and Theorem 3.3 complete the proof. □

Remark 3.1. For $s = 1/2$, the error estimates in Theorems 3.2 and 3.4 are the same as in [13], but our approach is different.

4. Numerical Experiments

In this section, we consider two examples to support theoretical analysis of Section 3. In both examples we use uniform meshes. In the first example the singular point x_i is far away from the endpoints and we choose $x_i = (b-a)/2$ — i.e. $i = (n+1)/2$, while in the second one the point x_i is very close to the endpoints and we set $i = n$.

Example 4.1. We use the nodal-type Newton-Cotes rules $\mathscr{Q}_h^{(k)}(x_{(n+1)/2})$ with $k = 1, 2, 3, 4$ to evaluate the integral

$$\fint_0^1 \frac{y^6}{|y - x_{(n+1)/2}|^{1+2s}}\, dy$$

for different singularities. Tables 1 and 3 show that the accuracy of the odd-order quadratures is $\mathscr{O}(h^{k+1}|\ln h|)$ for $s = 0$, $\mathscr{O}(h^k|\ln h|)$ for $s = 1/2$ and $\mathscr{O}(h^{k+1-2s})$ for other $s \in [0, 1)$. This agrees with the estimates of Theorem 3.2. On the other hand, Tables 2 and 4 show that the accuracy of the even-order rule is $\mathscr{O}(h^{k+2}|\ln h|)$ for $s = 0$, $\mathscr{O}(h^{k+1}|\ln h|)$ for $s = 1/2$ and $\mathscr{O}(h^{k+2-2s})$ for other $s \in [0, 1)$, one order higher than the general estimate in Theorem 3.2, which agrees with superconvergence estimate in Theorem 3.4. Observe that in this case, the factor $\eta_{(n+1)/2}$ is fixed as $1/2$ and does not influence the accuracy.

Example 4.2. We now use the nodal-type Newton-Cotes rules $\mathscr{Q}_h^{(k)}(x_n)$ with $k = 1, 2, 3, \cdots, 4$ to evaluate the integral

$$\fint_0^1 \frac{y^5}{|y - x_n|^{1+2s}}\, dy$$

for different singularities. Tables 5-8 show that if the singular point is close to an endpoint, the accuracy of each Newton-Cotes formula is $\mathscr{O}(h^{k+1}|\ln h|)$ for $s = 0$, $\mathscr{O}(h^k|\ln h|)$ for $s = 1/2$ and $\mathscr{O}(h^{k+1-2s})$ for other $s \in [0, 1)$, which agrees with the general estimate of Theorem 3.4. Note that now the superconvergence of the even-order rules disappears since $\eta_n = \mathscr{O}(h^{-1})$.

Appendix: Proof of Theorem 3.1

In order to prove Theorem 3.1, we need a result from [14], which is adjusted to the approach of the present work.

Proposition 4.1 (cf. Li [14]). *If $u \in C^{k+1}[r_0, r_k]$ and $\{r_j\}$ is a quasi-uniform mesh, then for any $\ell = 1, 2, \cdots, k$ one has*

$$\frac{u^{(\ell)}(r_0)}{\ell!} = \sum_{j=0}^{k} c_j^{(k,\ell)} u(r_j) + \mathscr{R}_1, \quad \frac{u^{(\ell)}(r_k)}{\ell!} = \sum_{j=0}^{k} \tilde{c}_j^{(k,\ell)} u(r_j) + \mathscr{R}_2, \tag{4.1}$$

Table 1: Errors of trapezoidal rule $\mathscr{Q}_h^{(1)}u(x_{(n+1)/2})$.

$n+1$	$s=0$		$s=1/4$		$s=1/2$		$s=3/4$	
2^8	5.287e-05		7.768e-04		4.670e-02		4.562e-01	
2^9	1.404e-05	1.91	2.730e-04	1.51	2.590e-02	0.85	3.227e-01	0.50
2^{10}	3.717e-06	1.92	9.613e-05	1.51	1.422e-02	0.86	2.282e-01	0.50
2^{11}	9.810e-07	1.92	3.389e-05	1.50	7.747e-03	0.88	1.613e-01	0.50
2^{12}	2.582e-07	1.93	1.196e-05	1.50	4.191e-03	0.89	1.141e-01	0.50

Table 2: Errors of Simpson's rule $\mathscr{Q}_h^{(2)}u(x_{(n+1)/2})$.

$n+1$	$s=0$		$s=1/4$		$s=1/2$		$s=3/4$	
2^8	4.330e-10		1.943e-08		2.467e-06		3.256e-05	
2^9	2.895e-11	3.90	1.722e-09	3.50	3.374e-07	2.87	5.755e-06	2.50
2^{10}	1.928e-12	3.91	1.525e-10	3.50	4.581e-08	2.88	1.017e-06	2.50
2^{11}	1.277e-13	3.92	1.350e-11	3.50	6.179e-09	2.89	1.798e-07	2.50
2^{12}	8.049e-15	3.99	1.196e-12	3.50	8.293e-10	2.90	3.180e-08	2.50

Table 3: Errors of Simpson's 3/8 rule $\mathscr{Q}_h^{(3)}u(x_{(n+1)/2})$.

$n+1$	$s=0$		$s=1/4$		$s=1/2$		$s=3/4$	
2^7	1.263e-09		4.771e-08		5.829e-06		7.901e-05	
2^8	8.339e-11	3.92	4.222e-09	3.50	7.974e-07	2.87	1.397e-05	2.50
2^9	5.492e-12	3.92	3.735e-10	3.50	1.083e-07	2.88	2.469e-06	2.50
2^{10}	3.612e-13	3.93	3.303e-11	3.50	1.461e-08	2.89	4.365e-07	2.50
2^{11}	2.359e-14	3.94	2.920e-12	3.50	1.960e-09	2.90	7.714e-08	2.50

Table 4: Errors of Cotes' rule $\mathscr{Q}_h^{(4)}u(x_{(n+1)/2})$.

$n+1$	$s=0$		$s=1/4$		$s=1/2$		$s=3/4$	
2^3	1.593e-07		2.770e-06		7.043e-05		4.250e-04	
2^4	2.644e-09	5.91	6.162e-08	5.49	2.510e-06	4.81	1.878e-05	4.50
2^5	4.371e-11	5.92	1.368e-09	5.49	8.810e-08	4.83	8.300e-07	4.50
2^6	7.205e-13	5.92	3.033e-11	5.50	3.056e-09	4.85	3.668e-08	4.50
2^7	1.177e-14	5.94	6.721e-13	5.50	1.049e-10	4.86	1.621e-09	4.50

Table 5: Errors of trapezoidal rule $\mathscr{Q}_h^{(1)}u(x_n)$.

$n+1$	$s=0$		$s=1/4$		$s=1/2$		$s=3/4$	
2^9	6.570e-05		2.532e-03		2.781e-01		3.469e+00	
2^{10}	1.757e-05	1.90	9.019e-04	1.49	1.529e-01	0.86	2.460e+00	0.50
2^{11}	4.673e-06	1.91	3.203e-04	1.49	8.332e-02	0.88	1.742e+00	0.50
2^{12}	1.238e-06	1.92	1.136e-04	1.50	4.507e-02	0.89	1.233e+00	0.50
2^{13}	3.268e-07	1.92	4.024e-05	1.50	2.423e-02	0.90	8.719e-01	0.50

Table 6: Errors of Simpson's rule $\mathcal{Q}_h^{(2)}u(x_n)$.

$n+1$	$s=0$		$s=1/4$		$s=1/2$		$s=3/4$	
2^9	5.645e-10		1.087e-08		7.351e-07		1.385e-05	
2^{10}	7.251e-11	2.96	2.118e-09	2.36	1.324e-07	2.47	3.541e-06	1.97
2^{11}	9.191e-12	2.98	3.916e-10	2.43	2.607e-08	2.34	1.012e-06	1.81
2^{12}	1.156e-12	2.99	7.074e-11	2.47	5.562e-09	2.23	3.154e-07	1.68
2^{13}	1.492e-13	2.95	1.256e-11	2.49	1.306e-09	2.09	1.055e-07	1.58

Table 7: Errors of Simpson's 3/8 rule $\mathcal{Q}_h^{(3)}u(x_n)$.

$n+1$	$s=0$		$s=1/4$		$s=1/2$		$s=3/4$	
2^7	1.293e-09		6.151e-08		7.721e-06		1.046e-04	
2^8	8.412e-11	3.94	5.465e-09	3.49	1.060e-06	2.86	1.856e-05	2.49
2^9	5.455e-12	3.95	4.844e-10	3.50	1.442e-07	2.88	3.288e-06	2.50
2^{10}	3.517e-13	3.96	4.289e-11	3.50	1.948e-08	2.89	5.817e-07	2.50
2^{11}	2.487e-14	3.82	3.794e-12	3.50	2.621e-09	2.89	1.030e-07	2.50

Table 8: Errors of Cotes' rule $\mathcal{Q}_h^{(4)}u(x_n)$.

$n+1$	$s=0$		$s=1/4$		$s=1/2$		$s=3/4$	
2^3	9.517e-09		2.929e-08		8.425e-08		2.363e-07	
2^4	3.244e-10	4.87	1.346e-09	4.44	5.356e-09	3.98	2.104e-08	3.49
2^5	1.052e-11	4.95	6.023e-11	4.48	3.360e-10	3.99	1.861e-09	3.50
2^6	3.340e-13	4.98	2.668e-12	4.50	2.100e-11	4.00	1.646e-10	3.50
2^7	1.155e-14	4.85	1.172e-13	4.51	1.279e-12	4.04	1.432e-11	3.52

where

$$c_j^{(k,\ell)} = \begin{cases} (r_j - r_0)^{-1}, & k=1 \text{ and } j=1, \\ \dfrac{1}{\omega_k'(r_j)}\sigma_{k-\ell}^{k-1}(\underbrace{r_0-r_1,\cdots,r_0-r_k}_{\text{neglect } r_0-r_j}), & k>1 \text{ and } j=1,\cdots,k, \end{cases}$$

$$\tilde{c}_j^{(k,\ell)} = \begin{cases} (r_k - r_j)^{-1}, & k=1 \text{ and } j=0, \\ \dfrac{1}{\omega_k'(r_j)}\sigma_{k-\ell}^{k-1}(\underbrace{r_k-r_0,\cdots,r_k-r_{k-1}}_{\text{neglect } r_k-r_j}), & k>1 \text{ and } j=0,\cdots,k-1, \end{cases}$$

$$c_0^{(k,\ell)} = -\sum_{j=1}^{k} c_j^{(k,\ell)}, \quad \tilde{c}_k^{(k,\ell)} = -\sum_{j=0}^{k-1} \tilde{c}_j^{(k,\ell)},$$

and $\omega_k(r)$, $\mathscr{R}_1$ *and* $\mathscr{R}_2$ *are, respectively, defined by* (3.5), (3.3) *and* (3.4).

Taking into account the proof of (3.2), we note that (4.1) will be proved if the following result holds.

Theorem 4.1. *If $u \in C^{k+1}[r_0, r_k]$, $\{r_j\}$ is a quasi-uniform mesh and $\Delta^{(k,\ell)}u(r_0)$, $\Delta^{(k,\ell)}u(r_k)$ are defined by* (3.1)*, then*

$$\Delta^{(k,\ell)}u(r_0) = \sum_{j=0}^{k} c_j^{(k,\ell)} u(r_j), \quad \Delta^{(k,\ell)}u(r_k) = \sum_{j=0}^{k} \tilde{c}_j^{(k,\ell)} u(r_j) \tag{4.2}$$

for any $\ell = 1, \cdots, k$.

We only prove the first equation in (4.2). The corresponding proof is based on the following lemmas.

Lemma 4.1. *If elementary symmetric polynomials are defined by* (2.1)*, then for any $\ell = 1, \cdots, m$ we have*

$$\begin{aligned}&(r_j - r_{m+1})\sigma_{m-\ell}^{m-1}(\underbrace{r_0 - r_1, \cdots, r_0 - r_m}_{neglect\ r_0 - r_j}) + \sigma_{m+1-\ell}^{m}(r_0 - r_1, \cdots, r_0 - r_m)\\ &= \sigma_{m+1-\ell}^{m}(\underbrace{r_0 - r_1, \cdots, r_0 - r_{m+1}}_{neglect\ r_0 - r_j}), \quad j = 1, \cdots, m.\end{aligned}$$

The proof of this result follows immediately from the relation

$$\begin{aligned}&\sigma_{m+1-\ell}^{m}(r_0 - r_1, \cdots, r_0 - r_m)\\ &= (r_0 - r_j)\sigma_{m-\ell}^{m-1}(\underbrace{r_0 - r_1, \cdots, r_0 - r_m}_{neglect\ r_0 - r_j}) + \sigma_{m+1-\ell}^{m-1}(\underbrace{r_0 - r_1, \cdots, r_0 - r_m}_{neglect\ r_0 - r_j}).\end{aligned}$$

Lemma 4.2. *If elementary symmetric polynomials are defined by* (2.1)*, then for any $\ell = 1, \cdots, m$ we have*

$$\sigma_{k-\ell}^{k-1}(\underbrace{r_0 - r_1, \cdots, r_0 - r_k}_{neglect\ r_0 - r_j}) = \sum_{m=\max\{\ell, j\}}^{k} \beta_{mj}^{(k)} \sigma_{m-\ell}^{m-1}(r_0 - r_1, \cdots, r_0 - r_{m-1}), \quad j = 1, \cdots, k, \tag{4.3}$$

where

$$\beta_{mj}^{(k)} = \begin{cases} 1, & m = k,\\ \prod_{j'=m+1}^{k}(r_j - r_{j'}), & \max\{\ell, j\} \le m \le k-1. \end{cases} \tag{4.4}$$

Proof. We prove the representation (4.3) by the method of mathematical induction.

The Eq. (4.3) is obviously true for $k = 2$. Assume that it is true for $k = k'$. The induction assumption and Lemma 4.1 show that for any $j = 1, \cdots, k'$ we have

$$\begin{aligned}&\sum_{m=\max\{j,\ell\}}^{k'+1} \beta_{mj}^{(k'+1)} \sigma_{m-\ell}^{m-1}(r_0 - r_1, \cdots, r_0 - r_{m-1})\\ &= (r_j - r_{k'+1}) \sum_{m=\max\{j,\ell\}}^{k'} \beta_{mj}^{(k')} \sigma_{m-\ell}^{m-1}(r_0 - r_1, \cdots, r_0 - r_{m-1}) + \sigma_{k'+1-\ell}^{k'}(r_0 - r_1, \cdots, r_0 - r_{k'})\end{aligned}$$

$$
\begin{aligned}
&=(r_j-r_{k'+1})\sigma_{k'-\ell}^{k'-1}(\underbrace{r_0-r_1,\cdots,r_0-r_{k'}}_{neglect\ r_0-r_j})+\sigma_{k'+1-\ell}^{k'}(r_0-r_1,\cdots,r_0-r_{k'})\\
&=\sigma_{k'+1-\ell}^{k'}(\underbrace{r_0-r_1,\cdots,r_0-r_{k'+1}}_{neglect\ r_0-r_j}).
\end{aligned}
$$

It is clear that (4.3) also holds for $j=k'+1$. □

The Proof of Theorem 4.1. Using the relation

$$u[r_0,r_1,\cdots,r_m]=\sum_{j=0}^{m}\frac{u(r_j)}{\omega_m'(r_j)},\quad \omega_m(r)=\prod_{j'=0}^{m}(r-r_{j'}),$$

we represent the term $\Delta^{(k,\ell)}u(r_0)$ in (3.1) in the form

$$\Delta^{(k,\ell)}u(r_0):=\sum_{j=0}^{k}d_j^{(k,\ell)}u(r_j),$$

where

$$d_j^{(k,\ell)}=\frac{1}{\omega_k'(r_j)}\sum_{m=\max\{\ell,j\}}^{k}\frac{\omega_k'(r_j)}{\omega_m'(r_j)}\sigma_{m-\ell}^{m-1}(r_0-r_1,\cdots,r_0-r_{m-1}),\quad j=0,1,\cdots,k.$$

Moreover, it follows from $\omega_k'(r_j)/\omega_m'(r_j)=\beta_{mj}^{(k)}$ that

$$d_j^{(k,\ell)}=\frac{1}{\omega_k'(r_j)}\sum_{m=\max\{\ell,j\}}^{k}\beta_{mj}^{(k)}\sigma_{m-\ell}^{m-1}(r_0-r_1,\cdots,r_0-r_{m-1}),$$

with the coefficients $\beta_{mj}^{(k)}$ described in (4.4). Proposition 4.1 and Lemma 4.2 yield

$$d_j^{(k,\ell)}=c_j^{(k,\ell)},\quad j=1,\cdots,k.$$

Moreover, using the representations (3.1) with $u(t)\equiv 1$, we obtain

$$\sum_{j=0}^{k}d_j^{(k,\ell)}=0.$$

Therefore,

$$d_0^{(k,\ell)}=-\sum_{j=1}^{k}d_j^{(k,\ell)}=-\sum_{j=1}^{k}c_j^{(k,\ell)}=c_0^{(k,\ell)},$$

and the proof is completed. □

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