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An accurate and asymptotically compatible collocation scheme for nonlocal diffusion problems ☆

Xiaoping Zhang [a], Jiming Wu [b], Lili Ju [c,d,*]

[a] School of Mathematics and Statistics, Wuhan University, Wuhan, Hubei 430072, PR China
[b] Institute of Applied Physics and Computational Mathematics, P.O. Box 8009, Beijing 100088, PR China
[c] Department of Mathematics, University of South Carolina, Columbia, SC 29208, USA
[d] Beijing Computational Science Research Center, Beijing 100193, PR China

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ABSTRACT

In this paper, we develop and analyze a collocation scheme for solving the linear nonlocal diffusion problem with general kernels. To approximate the nonlocal diffusion operator, we take a classic trapezoidal rule based on the linear interpolation as the starting point, and then carefully derive a new improved quadrature rule, which is not only more accurate but also could avoid the evaluations of singular integrals. We then use this rule to construct a collocation scheme for solving the nonlocal diffusion equations, that produces a symmetric positive definite stiffness matrix with Toeplitz structure. The proposed scheme is rigorously shown to be of second order accurate with respect to the mesh size for the nonlocal problem with fixed horizon, and in particular, it can achieve higher order accuracy for the commonly used kernels in the literature. Furthermore, we also prove that the scheme is asymptotically compatible, i.e., the approximate solution of the nonlocal diffusion problem converges to the exact solution of the corresponding local PDE problem when the horizon and the mesh size both go to zero. Finally, numerical experiments are presented to verify the theoretical results.

1. Introduction

Nonlocal phenomenon are ubiquitous in nature and nonlocal models have appeared in many subjects, from physics and biology to materials and social sciences. For example, there has been a great deal of interest recently in the nonlocal peridynamics (PD) continuum theory introduced first by Silling [21–25]. PD model is an integral-type nonlocal continuum model for the mechanics of materials, which provides an alternative setup to that of classic continuum mechanics based on partial differential equations (PDEs). Linear scalar PD operators also share similarities with nonlocal diffusion operators, as pointed out in [8], thus making the study of PD relevant to the study of general nonlocal diffusion (ND) models in various applications [2–5,13–15,17,19]. Mathematical analysis of PD and ND models can be found in [5,7–9,11–13]. Various discretization methods have also been proposed for the spatial nonlocal PD models, including finite element [1,6,10,16,20, 26,28,29,33], finite difference [26], collocation [30–32] and meshfree methods [18].

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* Corresponding author at: Department of Mathematics, University of South Carolina, Columbia, SC 29208, USA.
E-mail addresses: xpzhang.math@whu.edu.cn (X. Zhang), wu_jiming@iapcm.ac.cn (J. Wu), ju@math.sc.edu (L. Ju).

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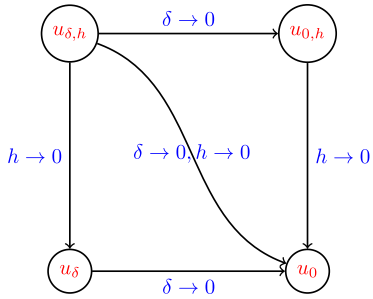

Fig. 1. A diagram for asymptotically compatible schemes and convergence results.

It is well-known that a common feature of PD and ND models is the introduction of the horizon parameter δ that characterizes the range of nonlocal interactions [9,21]. As $\delta \to 0$, the nonlocal effect diminishes and the zero-horizon limit of nonlocal PD models becomes a classical local PDE model when the latter is well-defined. Such limiting behavior provides connections and consistencies between nonlocal and local models, and has immense practical significance especially for multiscale modeling and simulations. A natural question is how such limiting behaviors can be preserved in various discrete approximations. This is a critical issue in the applications of PD like models to problems involving possibly different scales, given the popularity and practicality to perform PD simulations with a coupled horizon δ and mesh spacing h. Recently, for certain classes of parametrized problems, Du et al. [27] have introduced the concept of *asymptotically compatible* schemes and also established an abstract mathematical framework for their rigorous numerical analysis. For completeness, we here give an exposition of this concept and take the ND model as an example. Let u_δ and u_0 be the exact solution of the ND model and its limiting local PDE model, respectively, and let $u_{\delta,h}$ and $u_{0,h}$ be their corresponding discretization solutions. While we have the convergence of $\{u_{\delta,h}\}$ for a given δ as $h \to 0$, as well as the convergence of u_δ to u_0 and $u_{\delta,h}$ to $u_{0,h}$ as $\delta \to 0$, we are also interested in the behavior as both $\delta \to 0$ and $h \to 0$. So the following definition is given.

Definition 1. [27] A family of convergent approximations $\{u_{\delta,h}\}$ is said to be asymptotically compatible to the solution u_0 if as $\delta \to 0$ and $h \to 0$, we have $u_{\delta,h} \to u_0$.

It can be seen from Fig. 1 that an asymptotically compatible scheme can not only ensure to solve the ND model well if $\delta > 0$ but also obtain an approximation solution of the limiting local differential equation model as $\delta \to 0$. Therefore, the investigation of the asymptotically compatible schemes has directive function for the development of discretization schemes in the numerical simulation of ND models. The major contribution of this work is to develop an asymptotically compatible collocation scheme for solving the ND models.

As we pointed out before, the finite element discretization has been widely investigated for PD and ND models. But being based on variational formulations, it runs up against the embarrassment of the generation of stiffness matrices, and the difficulty of computing the entries become particular evident in higher dimension, due to the higher-order singularity of the inner and outer integrals as well as the irregular domains of integrations. An alternative way is to seek collocation discretization schemes due to their simplicity of implementation. In this work, we start with a classic trapezoidal rule based on the piecewise linear interpolation and then develop an improved quadrature rule for approximating the nonlocal operator, with their optimal error estimates being derived. This new rule also often could avoid the evaluations of singular integrals. Then based on this rule, we construct a collocation discretization scheme for solving nonlocal diffusion problems. By analyzing the stiffness matrices we show that the proposed scheme produces a symmetric positive definite discrete system with Toeplitz structure and satisfies the discrete maximum principle, and then starting from this, the following conclusions are further rigorously obtained: (1) the scheme is of second order convergent for the nonlocal diffusion problem with general kernels when the horizon is fixed; (2) if the kernel of the nonlocal operator is specified by a commonly used form in the literature, the convergence order is even higher (up to 3), which shows superiority compared to the quadrature-based finite difference method developed in [26]; (3) the scheme is also asymptotically compatible.

The rest of the paper is organized as follows. In Section 2, we first describe the nonlocal diffusion problem and give some useful notations. Two quadrature rules for approximating the nonlocal operator, including their formulation and error estimates, are discussed in Section 3. In Section 4, a collocation scheme is proposed for solving the nonlocal diffusion problem, along with some properties of the discretized system be discussed. In Section 5, the convergence and error estimates of the proposed scheme are proved, and we also show that the scheme is asymptotically compatible. Results of numerical experiments are reported in Section 6 to complement the theoretical analysis. Some conclusions are given in Section 7.

2. Mathematical model and some notations

Let Ω be a finite bar in $\mathbb{R}$. Without loss of generality, we take $\Omega = (0, 1)$. Let $\delta > 0$ be a horizon parameter. A nonlocal diffusion operator $\mathcal{L}_\delta$ is defined as, for any function $u = u(x) : \Omega \to \mathbb{R}$,

$$\mathcal{L}_\delta u(x) = \int\limits_{B_\delta(x)} \gamma_\delta(x,y)[u(y)-u(x)]\,dy, \quad \forall x \in \Omega \tag{1}$$

with $B_\delta(x) = \{y \in \mathbb{R} : |y-x| \le \delta\}$ denoting a neighborhood centered at x of radius δ, and $\gamma_\delta(x,y) : \Omega \times \Omega \to \mathbb{R}$ being a symmetric nonlocal kernel, i.e., $\gamma_\delta(x,y) = \gamma_\delta(y,x)$, and $\gamma_\delta(x,y) = 0$ if $y \notin B_\delta(x)$. The following nonlocal constrained value problem on a nonzero measure volume is our main subject of interest:

$$\begin{cases} -\mathcal{L}_\delta u(x) = f(x), & \text{in } \Omega, \\ \quad u(x) = g(x), & \text{on } \Omega_{\mathcal{I}}, \end{cases} \tag{2}$$

where $\Omega_{\mathcal{I}} = [-\delta, 0] \cup [1, 1+\delta]$ denotes the constrained domain. The constraint in (2) is a natural extension of Dirichlet boundary condition for classic PDEs.

We introduce the following assumptions for the kernel γ_δ:

(A1) $\gamma_\delta(x,y)$ is nonnegative and has the form of $\gamma_\delta(|y-x|)$;
(A2) the second order moment of γ_δ is positive and finite, i.e.,

$$0 < C_\delta = \int_0^\delta \gamma_\delta(z) z^2\,dz < \infty; \tag{3}$$

(A3) γ_δ is decreasing, i.e., $\gamma_\delta'(z) \le 0$ if $z \in (0, \delta]$.

The assumption **(A2)** is the most general condition for a well-defined elastic modulus or a well-defined diffusion constant. Such condition allows for more general kernels that may or may not be locally integrable. By using the symmetry of $B_\delta(x)$ and $\gamma_\delta(x,y)$ as well as the assumption **(A1)**, the nonlocal operator (1) can be rewritten as

$$\mathcal{L}_\delta u(x) = \int_0^\delta \gamma_\delta(z)\,[u(x-z) - 2u(x) + u(x+z)]\,dz. \tag{4}$$

For the commonly used kernel in the literature [6,26,27,32]

$$\gamma_\delta(z) = \frac{2-2s}{\delta^{2-2s}} z^{-1-2s}, \quad s \in [-0.5, 1), \tag{5}$$

it can be easily verified that it satisfies the assumptions **(A1)**, **(A2)** and **(A3)** with $C_\delta = 1$.

For simplicity, with $\Omega = (0,1)$, we consider a uniform mesh throughout our discussion. For a positive integer n, we set $h = 1/(n+1)$ and let $\delta = \kappa h + \delta_0$ for a nonnegative integer $\kappa < n$ and $\delta_0 \in (0, h]$. We here emphasize the relationship between δ, h and κ, which says that

- $\kappa = 0$ means $\delta \le h$;
- $\kappa \ge 1$ means $\delta > h$.

For the clarity of statement, we need to define a partition of $[-\delta, \delta]$, i.e.,

$$I_j = \begin{cases} [-\delta, -\kappa h], & j = -\kappa, \\ [(j-1)h, jh], & -\kappa+1 \le j \le \kappa \\ [\kappa h, \delta], & j = \kappa+1. \end{cases}$$

We now introduce grid points on $\Omega \cup \Omega_{\mathcal{I}}$ as $\{x_i = ih\}_{i \in \mathcal{N}}$, where the index set is defined by $\mathcal{N} = \{-\kappa, \cdots, 0, 1, \cdots, n+\kappa+1\}$. We also define the index sets

$$\mathcal{N}_{in} = \{1, 2, \cdots, n\} \text{ and } \mathcal{N}_{out} = \mathcal{N} \backslash \mathcal{N}_{in}, \tag{6}$$

which are corresponding to the interior and exterior grid points, respectively. We consider the discrete function space given by $\mathbb{V}_h = \{u_{\delta,h}(x_i), i \in \mathcal{N}_{in}\}$, which is isomorphism to $\mathbb{R}^n$. It is more convenient to use sub-index i for the discrete function: $u_i := u_{\delta,h}(x_i)$. Note that the value of a discrete function is only defined at grid points. Values inside each subinterval can be obtained by interpolation of values at grid points. In order to analyze the error later, we need to put the problem into a norm space. A "natural" norm for the finite linear space $\mathbb{V}_h$ is the maximum norm: for $u_{\delta,h} \in \mathbb{V}_h$,

$$\|u_{\delta,h}\|_\infty = \max_{i \in \mathcal{N}_{in}} |u_{\delta,h}(x_i)|.$$

Before the statement of the numerical schemes, we define

$$\Theta_k(a,b) = \int_a^b \gamma_\delta(z) z^k \, dz, \qquad k = 0, 1, \cdots, 4, \tag{7}$$

which will be frequently used in the subsequent analysis. We specially remark that, for the specific kernel $\gamma_\delta(z)$ (5), we have

$$\begin{aligned}
&\Theta_0(h,\delta) = \tfrac{2-2s}{\delta^{2-2s}} \begin{cases} \ln \frac{\delta}{h}, & s = 0, \\ \frac{\delta^{-2s} - h^{-2s}}{-2s}, & \text{otherwise}, \end{cases} \\
&\Theta_1(0,b) = \tfrac{2-2s}{\delta^{2-2s}} \begin{cases} \ln b, & s = 0.5, \\ \frac{b^{1-2s}}{1-2s}, & \text{otherwise}, \end{cases} \\
&\Theta_1(h,\delta) = \tfrac{2-2s}{\delta^{2-2s}} \begin{cases} \ln \frac{\delta}{h}, & s = 0.5, \\ \frac{\delta^{1-2s} - h^{1-2s}}{1-2s}, & \text{otherwise}, \end{cases} \\
&\Theta_2(0,\delta) = C_\delta = 1, \quad \Theta_2(0,h) = \tfrac{h^{2-2s}}{\delta^{2-2s}}, \\
&\Theta_4(0,\delta) = \tfrac{1-s}{2-s}\delta^2, \quad \Theta_4(0,h) = \tfrac{1-s}{2-s} \cdot \tfrac{h^{4-2s}}{\delta^{2-2s}}.
\end{aligned} \tag{8}$$

All the above computations are straightforward except for $\Theta_1(0,b)$. The computation of $\Theta_1(0,b)$ reduces to a singular integral $\int_0^b z^{-2s}\,dz$. If $s < 0.5$, it is a weakly singular integral and can be integrated normally; but if $s \geq 0.5$, it should be understood in Hadamard finite-part sense, which says that

$$\int_0^b z^{-2s}\,dz = \lim_{\epsilon \to 0} \begin{cases} \displaystyle\int_\epsilon^b z^{-1}\,dz + \ln \epsilon, & s = 0.5, \\ \displaystyle\int_\epsilon^b z^{-2s}\,dz + \frac{\epsilon^{1-2s}}{1-2s}, & s > 0.5. \end{cases}$$

3. Quadrature rules for approximating the nonlocal operator and their error estimates

In this section, we will propose some quadrature rules for approximating the nonlocal operator (1) and then derive their error estimates.

3.1. A classic quadrature rule for the nonlocal operator

Denote by

$$\Pi_h u(x) = \sum_{j=-\kappa}^{n+\kappa+1} u(x_j)\phi_j(x) \tag{9}$$

the piecewise-linear interpolation of $u(x)$, where $\phi_j(x)$ is the standard continuous piecewise linear *hat* basis function defined by

$$\phi_j(x) = \begin{cases} (x - x_{j-1})/h, & x \in (x_{j-1}, x_j], \\ (x_{j+1} - x)/h, & x \in (x_j, x_{j+1}), \\ 0, & \text{otherwise}. \end{cases} \tag{10}$$

Substituting (9) into (4) at the interior nodal points x_i $(i \in \mathcal{N}_{in})$ and noticing the fact that $\phi_{i+m}(x_i \pm z) = \phi_m(\pm z)$ yield the classic quadrature (trapezoidal rule based on the linear interpolation) rule:

$$\mathcal{L}_{\delta,h}^{(1)}u(x_i) = \int_0^\delta \gamma_\delta(z)\left[\sum_{m=-(\kappa+1)}^{\kappa+1} u(x_{i+m})\,(\phi_m(-z)+\phi_m(z)) - 2u(x_i)\right]dz$$
$$:= \sum_{m=-(\kappa+1)}^{\kappa+1} \alpha_m^{(1)} u(x_{i+m}), \quad i\in\mathcal{N}_{in}, \tag{11}$$

where

$$\alpha_m^{(1)} = \begin{cases} 2\int_{I_1} \gamma_\delta(z)\phi_m(z)\,dz - 2\int_0^\delta \gamma_\delta(z)\,dz, & m=0,\\ \int_{I_m\cup I_{m+1}} \gamma_\delta(z)\phi_m(z)\,dz, & m=1,\cdots,\kappa,\\ \int_{I_{\kappa+1}} \gamma_\delta(z)\phi_{\kappa+1}(z)\,dz, & m=\kappa+1, \end{cases} \tag{12}$$

and $\alpha_{-m}^{(1)} = \alpha_m^{(1)}$. It should be noted that when $\kappa=0$, the second line of (12) vanishes and $\alpha_m^{(1)}$ reduces to

$$\alpha_m^{(1)} = h^{-1}\begin{cases} -2\Theta_1(0,\delta), & m=0,\\ \Theta_1(0,\delta), & m=1, \end{cases} \tag{13}$$

and when $\kappa>0$, we have

$$\alpha_m^{(1)} = h^{-1}\begin{cases} -2\Theta_1(0,h) - 2h\int_{x_1}^\delta \gamma_\delta(z)\,dz, & m=0,\\ \Theta_1(0,h) + \int_{I_2}\gamma_\delta(z)(x_2-z)\,dz, & m=1,\\ \int_{I_m}\gamma_\delta(z)(z-x_{m-1})\,dz + \int_{I_{m+1}}\gamma_\delta(z)(x_{m+1}-z)\,dz, & m=2,\cdots,\kappa,\\ \int_{I_{\kappa+1}}\gamma_\delta(z)(z-x_\kappa)\,dz, & m=\kappa+1. \end{cases} \tag{14}$$

Note that the quadrature rule (11) is exact for any linear function. As a consequence, by choosing $u(x)\equiv 1$ we can easily get

$$\sum_{m=-(\kappa+1)}^{\kappa+1} \alpha_m^{(1)} = 0,$$

from which an equivalent formulation of the quadrature rule (11) is obtained, namely,

$$\mathcal{L}_{\delta,h}^{(1)}u(x_i) = \sum_{m=1}^{\kappa+1} \alpha_m^{(1)}[u(x_{i-m}) - 2u(x_i) + u(x_{i+m})]. \tag{15}$$

In this expression, there is no need to define $\alpha_0^{(1)}$, since this term makes no contribution to the sum.

Remark 1. In the sense of (15), $\mathcal{L}_{\delta,h}^{(1)}u(x_i)$ also can be regarded as a *quadrature-based finite difference approximation* for (4), however as demonstrated in [26] such scheme is not asymptotically compatible when used for solving the nonlocal diffusion problem (2).

We next will prove an error estimate for the quadrature rule (11).

Theorem 1. *Assume that $u \in C^4(\Omega)$. Then for the quadrature rule (11), it holds that*

$$\mathcal{L}_\delta u(x_i) - \mathcal{L}_{\delta,h}^{(1)} u(x_i) = -h^{-1}\Theta_1(0,\zeta)\,[u(x_{i-1}) - 2u(x_i) + u(x_{i+1})] + \sum_{m=-\kappa}^{\kappa} \omega_m u''(x_{i+m}) + \mathcal{R}_i(u), \tag{16}$$

for $i \in \mathcal{N}_{in}$, where $\zeta = \min(\delta, h)$,

$$|\mathcal{R}_i(u)| \le \begin{cases} C\Theta_4(0,\delta), & \kappa = 0, \\ C\left[\Theta_4(0,h) + h^4\Theta_0(h,\delta) + h^3\Theta_1(h,\delta)\right], & \kappa \ge 1, \end{cases} \tag{17}$$

$$\omega_0 = \Theta_2(0,\zeta) = \begin{cases} \Theta_2(0,\delta), & \kappa = 0, \\ \Theta_2(0,h), & \kappa \ge 1, \end{cases} \tag{18}$$

and for $\kappa \ge 1$,

$$\omega_{-m} = \omega_m = \frac{1}{2}\int_{I_{m+1}} \gamma_\delta(z)(z - x_m)(z - x_{m+1})\,dz, \quad m = 1, \cdots, \kappa. \tag{19}$$

Proof. Define the error function $e(x) = u(x) - \Pi_h u(x)$. We can split the error as

$$\mathcal{L}_\delta u(x_i) - \mathcal{L}_{\delta,h}^{(1)} u(x_i) = \mathcal{E}_1 + \mathcal{E}_2,$$

where

$$\mathcal{E}_1 = \int_0^\zeta \gamma_\delta(z)[e(x_i + z) + e(x_i - z)]\,dz,$$

$$\mathcal{E}_2 = \int_\zeta^\delta \gamma_\delta(z)[e(x_i + z) + e(x_i - z)]\,dz.$$

Now let us estimate $\mathcal{E}_i$ $(i = 1, 2)$ term by term. We first estimate $\mathcal{E}_1$. By using Taylor expansion of $u(x)$ at the point x_i, we have for $z \in [0, \zeta]$,

$$\begin{aligned} e(x_i - z) &= \left(-u'(x_i) - \tfrac{u(x_{i-1}) - u(x_i)}{h}\right) z + \tfrac{u''(x_i)}{2!} z^2 - \tfrac{u'''(x_i)}{3!} z^3 + \tfrac{u^{(4)}(\xi_1)}{4!} z^4, \\ e(x_i + z) &= \left(u'(x_i) - \tfrac{u(x_{i+1}) - u(x_i)}{h}\right) z + \tfrac{u''(x_i)}{2!} z^2 + \tfrac{u'''(x_i)}{3!} z^3 + \tfrac{u^{(4)}(\xi_2)}{4!} z^4, \end{aligned}$$

where $\xi_1 \in [x_i - z, x_i]$, $\xi_2 \in [x_i, x_i + z]$. Then, through some straightforward calculations, we have

$$\mathcal{E}_1 = \omega_0 u''(x_i) - h^{-1}\Theta_1(0,\zeta)\,[u(x_{i-1}) - 2u(x_i) + u(x_{i+1})] + \mathcal{E}_{11}, \tag{20}$$

where ω_0 is defined in (18) and

$$\mathcal{E}_{11} = \frac{1}{4!}\int_0^\zeta \left[u^{(4)}(\xi_1) + u^{(4)}(\xi_2)\right] \gamma_\delta(z) z^4\,dz.$$

From the assumption **(A2)** we see that $\gamma_\delta(z)z^4$ is integrable on $[0, \zeta]$, and then under the assumption $u \in C^4(\Omega)$ we have

$$|\mathcal{E}_{11}| \le C\Theta_4(0,\zeta).$$

Meanwhile, we should note that if $\kappa = 0$, it holds that

$$|\mathcal{R}_i(u)| = |\mathcal{E}_{11}| \le C\Theta_4(0,\delta).$$

Next, we estimate $\mathcal{E}_2$. Note that $\mathcal{E}_2$ has no singularity and it vanishes in the case of $\kappa = 0$, so we only need to consider the case of $\kappa \ge 1$. Applying Taylor expansion of $u(x_i + z), z \in I_{m+1}$ and $u(x_{i+m+1})$ at the point x_{i+m}, we get

$$u(x_i+z)-u(x_{i+m})=\sum_{l=1}^{3}\tfrac{u^{(l)}(x_{i+m})}{l!}(z-x_m)^l+\tfrac{u^{(4)}(\xi_3)}{4!}(z-x_m)^4,$$
$$u(x_{i+m+1})-u(x_{i+m})=\sum_{l=1}^{3}\tfrac{u^{(l)}(x_{i+m})}{l!}h^l+\tfrac{u^{(4)}(\xi_4)}{4!}h^4,$$

where $\xi_3\in[x_{i+m},x_i+z]$ and $\xi_4\in[x_{i+m},x_{i+m+1}]$. Hence,

$$\begin{aligned}e(x_i+z)&=u(x_i+z)-u(x_{i+m})-\frac{u(x_{i+m+1})-u(x_{i+m})}{h}(z-x_m)\\&=\frac{u''(x_{i+m})}{2!}(z-x_m)(z-x_{m+1})+\frac{u'''(x_{i+m})}{3!}(z-x_{m-1})(z-x_m)(z-x_{m+1})+\tilde{e}(x_i+z),\end{aligned}$$

where

$$\tilde{e}(x_i+z)=\frac{z-x_m}{4!}\left[u^{(4)}(\xi_3)(z-x_m)^3-u^{(4)}(\xi_4)h^3\right].\tag{21}$$

Similarly, for $z\in I_{m+1}$, we also have

$$e(x_i-z)=\frac{u''(x_{i-m})}{2!}(z-x_m)(z-x_{m+1})-\frac{u'''(x_{i-m})}{3!}(z-x_{m-1})(z-x_m)(z-x_{m+1})+\bar{e}(x_i-z),$$

where

$$\bar{e}(x_i-z)=\frac{z-x_m}{4!}\left[u^{(4)}(\xi_5)(z-x_m)^3-u^{(4)}(\xi_6)h^3\right]\tag{22}$$

with $\xi_5\in[x_i-z,x_{i-m}]$ and $\xi_6\in[x_{i-m-1},x_{i-m}]$. As a consequence, we get

$$\begin{aligned}e(x_i-z)+e(x_i+z)=&\frac{u''(x_{i+m})+u''(x_{i-m})}{2!}(z-x_m)(z-x_{m+1})\\&+\frac{u'''(x_{i+m})-u'''(x_{i-m})}{3!}(z-x_{m-1})(z-x_m)(z-x_{m+1})\\&+\tilde{e}(x_i+z)+\tilde{e}(x_i-z).\end{aligned}$$

Combining this with the definition of $\mathcal{E}_2$ and ω_m yields

$$\mathcal{E}_2=\sum_{m=1}^{\kappa}\omega_m[u''(x_{i-m})+u''(x_{i+m})]+\mathcal{E}_{21}+\mathcal{E}_{22}\tag{23}$$

with

$$\begin{aligned}\mathcal{E}_{21}&=\sum_{m=1}^{\kappa}\frac{u'''(x_{i+m})-u'''(x_{i-m})}{3!}\int\limits_{I_{m+1}}\gamma_\delta(z)(z-x_{m-1})(z-x_m)(z-x_{m+1})\,dz,\\\mathcal{E}_{22}&=\sum_{m=1}^{\kappa}\int\limits_{I_{m+1}}\gamma_\delta(z)\left[\tilde{e}(x_i-z)+\tilde{e}(x_i+z)\right]dz.\end{aligned}$$

For $\mathcal{E}_{21}$, by the assumption $u\in C^4(\Omega)$ we have

$$\begin{aligned}|\mathcal{E}_{21}|&\le C\sum_{m=1}^{\kappa}mh^4\int\limits_{I_{m+1}}\gamma_\delta(z)\,dz\\&\le Ch^3\sum_{m=1}^{\kappa}\left[\int\limits_{I_{m+1}}\gamma_\delta(z)(z-x_m)\,dz+\int\limits_{I_{m+1}}\gamma_\delta(z)z\,dz\right]\\&\le Ch^3\sum_{m=1}^{\kappa}\left[h\int\limits_{I_{m+1}}\gamma_\delta(z)\,dz+\int\limits_{I_{m+1}}\gamma_\delta(z)z\,dz\right]\\&=Ch^3\left[h\Theta_0(h,\delta)+\Theta_1(h,\delta)\right].\end{aligned}\tag{24}$$

For $\mathcal{E}_{22}$, we can first get

$$|\tilde{e}(x_i \pm z)| \le Ch^4, \quad z \in (x_m, x_{m+1}]$$

from (21) and (22), and then obtain that

$$|\mathcal{E}_{22}| \le Ch^4 \sum_{m=1}^{\kappa} \int_{I_{m+1}} \gamma_\delta(z)\,dz = Ch^4 \Theta_0(h,\delta). \tag{25}$$

Putting all these together concludes the proof. □

Based on the error estimate (16), the quadrature rule (11) will be used as the starting point to construct an improved and more accurate quadrature rule for the nonlocal diffusion operator (1).

3.2. An improved quadrature rule

Based on (16), we first define

$$\begin{aligned} \mathcal{L}_{\delta,h}^{(2)} u(x_i) &= \mathcal{L}_{\delta,h}^{(1)} u(x_i) - h^{-1}\Theta_1(0,\zeta)\,[u(x_{i-1}) - 2u(x_i) + u(x_{i+1})] \\ &:= \sum_{m=-(\kappa+1)}^{\kappa+1} \alpha_m^{(2)} u(x_{i+m}), \end{aligned} \tag{26}$$

where

$$\alpha_m^{(2)} = \alpha_m^{(1)} - \begin{cases} -2h^{-1}\Theta_1(0,\zeta), & m = 0, \\ h^{-1}\Theta_1(0,\zeta), & m = \pm 1, \\ 0, & \text{otherwise}. \end{cases} \tag{27}$$

We can directly obtain from Theorem 1 that

$$\mathcal{L}_\delta u(x_i) - \mathcal{L}_{\delta,h}^{(2)} u(x_i) = \sum_{m=-\kappa}^{\kappa} \omega_m u''(x_{i+m}) + \mathcal{R}_i(u). \tag{28}$$

Next further approximating $u''(x_{i+m})$ in (16) by the central difference scheme

$$\Delta_h u(x_{i+m}) = \frac{u(x_{i+m-1}) - 2u(x_{i+m}) + u(x_{i+m+1})}{h^2}, \tag{29}$$

then we finally get a new quadrature rule as

$$\begin{aligned} \mathcal{L}_{\delta,h} u(x_i) &= \begin{cases} \omega_0 \Delta_h u(x_i), & \kappa = 0, \\ \mathcal{L}_{\delta,h}^{(2)} u(x_i) + \sum_{m=-\kappa}^{\kappa} \omega_m \Delta_h u(x_{i+m}), & \kappa \ge 1 \end{cases} \\ &:= \sum_{m=-(\kappa+1)}^{\kappa+1} \alpha_m u(x_{i+m}), \end{aligned} \tag{30}$$

where

$$\alpha_m = \alpha_m^{(2)} + \sigma_m, \tag{31}$$

with

$$\sigma_m = h^{-2} \begin{cases} -2\omega_0, & m = 0, \\ \omega_0, & m = \pm 1, \end{cases} \tag{32}$$

when $\kappa = 0$ and

$$\sigma_m = h^{-2} \begin{cases} \omega_{-\kappa}, & m = -(\kappa+1), \\ \omega_{\kappa}, & m = \kappa+1, \\ -2\omega_{-\kappa} + \omega_{-\kappa+1}, & m = -\kappa, \\ \omega_{\kappa-1} - 2\omega_{\kappa}, & m = \kappa, \\ \omega_{m-1} - 2\omega_m + \omega_{m+1}, & \text{otherwise}, \end{cases} \tag{33}$$

when $\kappa \geq 1$. From the proof of Theorem 1 we can see that the residual $\mathcal{R}_i(u) \equiv 0$ if u is chosen to be an arbitrary quadratic function because

$$\mathcal{R}_i(u) = \begin{cases} \mathcal{E}_{11}, & \kappa = 0, \\ \mathcal{E}_{11} + \mathcal{E}_{21} + \mathcal{E}_{22}, & \kappa \geq 1 \end{cases}$$

can be formulated through the third and fourth derivatives of u, and moreover, the approximation (29) is obviously exact for arbitrary quadratic functions. Combination of these with (28) shows that $\mathcal{L}_{\delta,h}u(x_i)$ is exact for arbitrary quadratic functions. Thus $\mathcal{L}_{\delta,h}u(x_i)$ is an improved and higher-order accurate quadrature rule for (4). Similar to (15), $\mathcal{L}_{\delta,h}u(x_i)$ can be further reformulated as

$$\mathcal{L}_{\delta,h}u(x_i) = \sum_{m=1}^{\kappa+1} \alpha_m [u(x_{i-m}) - 2u(x_i) + u(x_{i+m})]. \tag{34}$$

Remark 2. Another big advantage of the quadrature rule $\mathcal{L}_{\delta,h}u(x_i)$ is that it could avoid the evaluations of singular integrals due to two facts: first, ω_0 in (18) is at most weakly singular and can be explicitly integrated in most cases, and ω_m defined by (19) for $|m| \geq 1$ has no singularity, so σ_m defined by (33) usually does not need evaluate singular integrals. Second, $\alpha_m^{(1)}$ for $|m| \leq 1$ defined by (13) or (14) may have weak or Hadamard finite-part singularity due to the existence of the factor $\Theta_1(0,\zeta)$, but this factor is canceled in $\alpha_m^{(2)}$ defined through (27), and thus $\alpha_m^{(2)}$'s are all regular.

Theorem 2. *Assume that $u \in C^4(\Omega)$. Then for the quadrature rule (30), it holds that*

$$|\mathcal{L}_\delta u(x_i) - \mathcal{L}_{\delta,h}u(x_i)| \leq \begin{cases} C\left[\Theta_4(0,\delta) + h^2\Theta_2(0,\delta)\right], & \kappa = 0, \\ C\left[\Theta_4(0,h) + h^2\Theta_2(0,h) + h^3\Theta_1(h,\delta) + h^4\Theta_0(h,\delta)\right], & \kappa \geq 1, \end{cases} \tag{35}$$

for $i = 1, \cdots, n$.

Proof. From Theorem 1 and the definition of $\mathcal{L}_{\delta,h}u(x_i)$, we see that

$$\mathcal{L}_\delta u(x_i) - \mathcal{L}_{\delta,h}u(x_i) = \sum_{m=-\kappa}^{\kappa} \omega_m \left[u''(x_{i+m}) - \Delta_h u(x_{i+m})\right] + \mathcal{R}_i(u),$$

where $\mathcal{R}_i(u)$ is estimated by (17). The result in the case of $\kappa = 0$ can be easily obtained, and thus we only need to consider the case that $\kappa \geq 1$. From the definition of ω_m in Theorem 1, we can see that

$$\omega_0 > 0, \;\; \omega_m < 0, \; m = 1, \cdots, \kappa$$

and

$$0 < -\sum_{m=-\kappa, m\neq 0}^{\kappa} \omega_m = -2\sum_{m=1}^{\kappa} \omega_m \leq 2h^2\Theta_0(h,\delta).$$

Since $\left|u''(x_{i+m}) - \Delta_h u(x_{i+m})\right| \leq Ch^2$, we have

$$\begin{aligned} &\left|\sum_{m=-\kappa}^{\kappa} \omega_m \left[u''(x_{i+m}) - \Delta_h u(x_{i+m})\right]\right| \\ &\leq Ch^2 \sum_{m=-\kappa}^{\kappa} |\omega_m| = Ch^2 \left(\omega_0 - \sum_{m=-\kappa, m\neq 0}^{\kappa} \omega_m\right) \\ &\leq Ch^2 \left[\Theta_2(0,h) + 2h^2\Theta_0(h,\delta)\right]. \end{aligned} \tag{36}$$

The proof is completed by combining the estimation for $\mathcal{R}_i(u)$ (17) with the above inequality (36). $\square$

4. A collocation discretization method for the nonlocal diffusion problem

With the quadrature rule (30), we propose a collocation scheme for discretizing the nonlocal diffusion problem (2) as follows: find $u_{\delta,h} \in \mathbb{V}_h$ such that

$$\begin{cases} -\mathcal{L}_{\delta,h} u_{\delta,h}(x_i) = f(x_i), & i \in \mathcal{N}_{in}, \\ u_{\delta,h}(x_i) = g(x_i), & i \in \mathcal{N}_{out}. \end{cases} \tag{37}$$

Let $\vec{U}$ be a column vector with entries $\{u_i = u_{\delta,h}(x_i)\}_{i\in\mathcal{N}_{in}}$, and $\vec{F}$ be that with entries $\{f_i\}_{i\in\mathcal{N}_{in}}$ being defined by

$$f_i = f(x_i) + \sum_{j\in\mathcal{N}_{out}} \alpha_{j-i} g(x_j),$$

we can rewrite the corresponding discrete linear system as

$$\mathbb{A}\vec{U} = \vec{F}, \tag{38}$$

where $\mathbb{A}$ is the nonlocal stiffness matrix with entries

$$a_{ij} = \begin{cases} -\alpha_{j-i}, & |j-i| \le \kappa + 1, \\ 0, & \text{otherwise.} \end{cases}$$

4.1. Properties of the nonlocal stiffness matrix

Next we will study the properties of the nonlocal matrix $\mathbb{A}$. First, we define an $n \times n$ matrix

$$\mathbb{A}^{(2)} = \left(a_{ij}^{(2)}\right), \quad a_{ij}^{(2)} = \begin{cases} -\alpha_{j-i}^{(2)}, & |j-i| \le \kappa + 1 \\ 0, & \text{otherwise,} \end{cases} \tag{39}$$

where $\alpha_{j-i}^{(2)}$ is defined in (27).

Lemma 1. *Let $\mathbb{A}^{(2)}$ be defined in (39), then $\mathbb{A}^{(2)}$ is a symmetric positive definite M-matrix with Toeplitz structure.*

Proof. The Toeplitz structure and symmetry of $\mathbb{A}^{(2)}$ is obvious from its definition in (39). To show it is an M-matrix, we can validate that it is a Z-matrix and all the eigenvalues are positive.

Firstly, let us check the sign symbol of $\mathbb{A}^{(2)}$. By checking their values through (27) and (12), the sign symbol of $\alpha_0^{(2)}$ and $\alpha_1^{(2)}$ can be determined to be negative and positive, respectively. Since $\alpha_m^{(2)} = \alpha_m^{(1)}, m \ge 2$, and from (12) we see that they are nonnegative because their integrands have no singularities and are always positive. Therefore, $\mathbb{A}^{(2)}$ is a Z-matrix. Secondly, let us verify that all eigenvalues of $\mathbb{A}^{(2)}$ are all positive. It is easy to see that $\sum_{j=1}^n a_{ij}^{(2)} \ge 0$, which says that $\mathbb{A}^{(2)}$ is a diagonally dominant matrix. By Gershgorin's circle theorem it holds

$$|\lambda - a_{ii}^{(2)}| \le -\sum_{j=1, j\ne i}^{n} a_{ij}^{(2)},$$

which leads to $\lambda \ge \sum_{j=1}^n a_{ij}^{(2)} \ge 0$. By noting that $\mathbb{A}^{(2)}$ is irreducible, we see that it is invertible, and therefore $\lambda \ne 0$, which means that λ is always positive. The proof is completed. $\square$

Note that the Toeplitz structure of the stiffness matrix will allow us to use the FFT based fast algorithm for matrix-vector multiplications in iterative linear system solvers for the problem.

Lemma 2. *For the sequence $\{\omega_m\}_{m=0}^{\kappa}$, we have $\omega_0 > 0$, $\omega_m < 0$ for $m = 1, \cdots, \kappa$. Furthermore, $\{\omega_m\}_{m=1}^{\kappa}$ constitutes an increasing sequence.*

Proof. $\omega_0 > 0$ is obvious by (8) and (18). Moreover, from (19), we see that $\omega_m < 0$ when $m \ge 1$. Now we start to prove the last result and always assume that $\kappa > 1$. By the definition of ω_m we have

$$\omega_{m-1}-\omega_m = \frac{1}{2}\begin{cases} \displaystyle\int_{I_{m+1}} [\gamma_\delta(z-h)-\gamma_\delta(z)]\,(z-x_m)(z-x_{m+1})\,dz, & m=2,\cdots,\kappa-1, \\ \displaystyle\int_{x_\kappa}^{\delta} [\gamma_\delta(z-h)-\gamma_\delta(z)]\,(z-x_\kappa)(z-x_{\kappa+1})\,dz & \\ \displaystyle\quad + \int_{\delta}^{(\kappa+1)h} \gamma_\delta(z-h)(z-x_\kappa)(z-x_{\kappa+1})\,dz, & m=\kappa, \end{cases} \tag{40}$$

then using the assumption **(A3)** we see that $\omega_{m-1}-\omega_m<0$ for $m=2,\cdots,\kappa$. □

Lemma 3. *For the sequence $\{\sigma_m\}_{m=0}^{\kappa+1}$ defined in (33), we have $\sigma_0<0$ and $\sigma_1>0$.*

Proof. The sign symbols of σ_0 and σ_1 are obvious in the case of $\kappa=0$. Now we consider the case of $\kappa>0$. From Lemma 2, we see that $\omega_0>0$ and $\omega_1<0$, and thus $h^2\sigma_0=-2\omega_0+2\omega_1<0$. Meanwhile, if $\kappa=1$, then $h^2\sigma_1=\omega_0-2\omega_1>0$; and if $\kappa>1$, then $h^2\sigma_1=\omega_0-\omega_1+(\omega_2-\omega_1)>0$, where we have used the last result in Lemma 2. □

Proposition 1. *For the collocation scheme (37), the nonlocal stiffness matrix $\mathbb{A}$ is a symmetric positive definite M-matrix with Toeplitz structure.*

Proof. From Lemma 1 and Lemma 3, we can easily obtain the symmetry and Toeplitz structure of $\mathbb{A}$. To show that $\mathbb{A}$ is an M-matrix, we can finish it by first showing it is a Z-matrix and then using the similar trick in Lemma 1. So it remains for us to check the sign symbols of $\alpha_m, m=0,1,\cdots,\kappa+1$. Also from Lemma 1 and Lemma 3, we see that $\alpha_0=\alpha_0^{(2)}+\sigma_0<0$ and $\alpha_1=\alpha_1^{(2)}+\sigma_1>0$. For $m=2,\cdots,\kappa+1$, α_m can be rewritten as

$$\alpha_m = \begin{cases} \alpha_m^{(2)}+h^{-2}\omega_{m-1}-h^{-2}\omega_m+h^{-2}(\omega_{m+1}-\omega_m), & m=2,\cdots,\kappa-2, \\ \alpha_\kappa^{(2)}+h^{-2}\omega_{\kappa-1}-2h^{-2}\omega_\kappa, & m=\kappa, \\ \alpha_{\kappa+1}^{(2)}+h^{-2}\omega_\kappa, & m=\kappa+1. \end{cases} \tag{41}$$

On the one hand, for $m=2,\cdots,\kappa$, we have

$$\alpha_m^{(2)}+h^{-2}\omega_{m-1} = \frac{h^{-2}}{2}\int_{I_m}\gamma_\delta(z)(z-x_{m-1})(z-x_{m-2})\,dz + h^{-1}\int_{I_{m+1}}\gamma_\delta(z)(x_{m+1}-z)\,dz>0 \tag{42}$$

and

$$\alpha_{\kappa+1}=\alpha_{\kappa+1}^{(2)}+h^{-2}\omega_\kappa = \frac{h^{-2}}{2}\int_{I_{\kappa+1}}\gamma_\delta(z)(z-x_{\kappa-1})(z-x_\kappa)\,dz>0. \tag{43}$$

On the other hand, from Lemma 2 we see that $\omega_m<0$ for $m=2,\cdots,\kappa$ and $\omega_{m+1}-\omega_m\ge 0$ for $m=2,\cdots,\kappa-2$. Combining these with (41), (42) and (43) concludes the proof. □

4.2. Discrete maximum principle

Since the kernels γ_δ of the nonlocal operators $\mathcal{L}_\delta$ is assumed to be nonnegative and symmetric in this work, it is easy to see that the solution of the nonlocal equation (2) satisfies the maximum principle. Based on the property of the nonlocal stiffness matrix, we can investigate whether such a property is preserved by the proposed collocation scheme in the discrete sense. Proposition 1 implies that $\mathbb{A}^{-1}$ is nonnegative, and the following discrete maximum principle (DMP) immediately follows.

Proposition 2. *The collocation scheme (37) satisfies the DMP:*

$$f(x)\le 0 \ \textit{for} \ x\in\Omega \Rightarrow \max_{i\in\mathcal{N}_{in}} u_i \le \max_{i\in\mathcal{N}_{out}}\{u_i\}$$

or

$$f(x) \geq 0 \text{ for } x \in \Omega \Rightarrow \min_{i \in \mathcal{N}_{in}} u_i \geq \min_{i \in \mathcal{N}_{out}} \{u_i\}.$$

The DMP can be readily used to establish the stability of the discrete scheme.

Proposition 3. *For the discrete operator $-\mathcal{L}_{\delta,h}$ defined by (30), it holds that $\|(-\mathcal{L}_{\delta,h})^{-1}\|_\infty$ is bounded as*

$$\|(-\mathcal{L}_{\delta,h})^{-1}\|_\infty \leq \frac{1+4\delta(1+\delta)}{8C_\delta}.$$

Proof. We choose $v_{\delta,h}(x) = \frac{x(1-x)+\delta(1+\delta)}{2C_\delta}$ and it is easy to see that $v_{\delta,h}(x_j) \geq 0$ for all $j \in \mathcal{N}_{out}$. Since $\mathcal{L}_{\delta,h}$ is exact for arbitrary quadratic functions, and thus we have

$$-\mathcal{L}_{\delta,h} v_{\delta,h} = (1, 1, \cdots, 1)^T.$$

By the DMP, it then holds that

$$\|(-\mathcal{L}_{\delta,h})^{-1}\|_\infty \leq \|v_{\delta,h}\|_\infty \leq \frac{1+4\delta(1+\delta)}{8C_\delta}.$$

The proof is completed. □

5. Convergence and error analysis of the collocation scheme

In this section, we investigate the convergence behavior and error estimates of the solution $u_{\delta,h}$ of the collocation scheme (37). The following theorem gives the general convergence result.

Theorem 3. *Assume that u_δ, the solution of nonlocal diffusion problem (2), belongs to $C^4(\Omega)$. For the collocation scheme (37), it holds that*

$$\|u_{\delta,h} - u_\delta\|_\infty \leq C\Theta_2^{-1}(0,\delta) \begin{cases} \left[\Theta_4(0,\delta) + h^2\Theta_2(0,\delta)\right], & \kappa = 0, \\ \left[\Theta_4(0,h) + h^2\Theta_2(0,h)\right. & \\ \quad \left. + h^3\Theta_1(h,\delta) + h^4\Theta_0(h,\delta)\right], & \kappa \geq 1. \end{cases} \tag{44}$$

Proof. It follows from (2) and (37) that

$$-\mathcal{L}_{\delta,h}[u_{\delta,h}(x_i) - u_\delta(x_i)] = -\mathcal{L}_\delta u_\delta(x_i) + \mathcal{L}_{\delta,h} u_\delta(x_i),$$

and hence,

$$\|u_{\delta,h} - u_\delta\|_\infty \leq \|(-\mathcal{L}_{\delta,h})^{-1}\|_\infty \| - \mathcal{L}_\delta u_\delta + \mathcal{L}_{\delta,h} u_\delta\|_\infty. \tag{45}$$

Based on Theorem 2 and Proposition 3, we can directly obtain (44) from (45). □

When the kernel function γ_δ is specified by (5), we see that $\Theta_2(0,\delta) = 1$ from (8). Then we can get the following corollary.

Corollary 1. *Under the same assumption of Theorem 3, and let the kernel be given by (5), it holds that*

$$\|u_{\delta,h} - u_\delta\|_\infty \leq C(\delta^2 + h^2) \tag{46}$$

for $\kappa = 0$ and

$$\|u_{\delta,h} - u_\delta\|_\infty \leq \begin{cases} C\left(\frac{h^4}{\delta^2} + \frac{h^4}{\delta^2}\left|\ln\frac{\delta}{h}\right| + \frac{h^3}{\delta}\right), & s = 0, \\ C\left(\frac{h^3}{\delta} + \frac{h^3}{\delta}\left|\ln\frac{\delta}{h}\right| + \frac{h^4}{\delta^2}\right), & s = 0.5, \\ C\left(\frac{h^{4-2s}}{\delta^{2-2s}} + \frac{h^4}{\delta^2} + \frac{h^3}{\delta}\right), & \text{otherwise}, \end{cases} \tag{47}$$

for $\kappa \geq 1$.

5.1. Convergence to the nonlocal problem with fixed horizon

Now we conclude the convergence of the collocation scheme (37) to the nonlocal problem with fixed horizon δ as the mesh size $h \to 0$. In this case, all nonlocal stiffness matrices are of the forms in the case that $\delta \geq h$, which means that $\kappa \geq 1$.

Theorem 4. *Assume that u_δ, the solution of nonlocal diffusion problem (2), belongs to $C^4(\Omega)$, and let the horizon $\delta > 0$ be fixed. Then it holds that*

$$\|u_{\delta,h} - u_\delta\|_\infty \leq Ch^2. \tag{48}$$

Proof. By using Assumption **(A2)** and the mean-value theorem of integrals we can get

$$\Theta_4(0,\delta) \leq \delta^2 \Theta_2(0,\delta), \quad \Theta_4(0,h) \leq h^2 \Theta_2(0,h).$$

Moreover, if $\kappa \geq 1$ we have

$$h\Theta_1(h,\delta) = \int_h^\delta \gamma_\delta(z) zh\, dz \leq \int_h^\delta \gamma_\delta(z) z^2\, dz = C_\delta,$$

$$h^2\Theta_0(h,\delta) = \int_h^\delta \gamma_\delta(z) h^2\, dz \leq \int_h^\delta \gamma_\delta(z) z^2\, dz = C_\delta.$$

Combining all these with the estimate (44) in Theorem 3 yields (48). □

Moreover, by setting a fixed δ in Corollary 1, we also obtain the following result.

Corollary 2. *Under the same assumption of Theorem 4, and let the kernel be given by (5), then it holds that*

$$\|u_{\delta,h} - u_\delta\|_\infty \leq \begin{cases} Ch^3|\ln h|, & s = 0.5, \\ Ch^{\min(3,4-2s)}, & \text{otherwise}. \end{cases} \tag{49}$$

Remark 3. Theorem 4 tells us that the collocation scheme (37) can achieve the second order convergence for the nonlocal diffusion problem with general kernels when the horizon δ is fixed. Furthermore, if the kernel function is specified by (5), Corollary 2 states that the accuracy of the scheme can be even higher, depending on the singularity parameter s of the kernel function. This fact shows the superiority of this collocation scheme compared to the quadrature-based finite difference scheme proposed in [26] which only has a convergence order of $O(h^2)$ for the kernel of form (5).

5.2. Asymptotic compatibility

To connect the nonlocal diffusion equation (2) with its local limit, we also require that

$$C_\delta \to C_0 > 0, \quad \text{as } \delta \to 0. \tag{50}$$

Under assumptions **(A1)** and **(A2)**, as $\delta \to 0$, the solution of nonlocal problems (2) converges to the solution of the two-point boundary value problem [8,34]:

$$\begin{cases} -C_0 u''(x) = f(x), & \text{in } \Omega, \\ u(x) = g(x), & \text{on } \partial\Omega = \{0\} \cup \{1\}, \end{cases} \tag{51}$$

which is the classic diffusion problem.

To study the asymptotic compatibility of the proposed collocation scheme (37), i.e., convergence of $u_{\delta,h}$ to the exact solution u_0 of the corresponding local diffusion problem (51) as both the mesh size δ and the horizon δ goes to zero, we first consider the limit of $\mathcal{L}_{\delta,h}$ with fixed h as $\delta \to 0$. In this case, all nonlocal stiffness matrices are falling into the case of $\delta \leq h$ (which means that $\kappa = 0$), thus we see from (18) and (30) that

$$\mathcal{L}_{\delta,h}(x_i) = \omega_0 \frac{u(x_{i-1}) - 2u(x_i) + u(x_{i+1})}{h^2},$$

where $\omega_0 = \Theta_2(0,\delta) = C_\delta$. Connecting it with the condition (50) yields the following result.

Theorem 5. *For the collocation scheme (37) with fixed h for the nonlocal diffusion problem (2), we have $C_\delta \mathbb{A} \to C_0 \mathbb{A}$ as $\delta \to 0$, and so the scheme converges to the standard central finite difference scheme for the corresponding local problem (51).*

Suppose that u_0 is smooth enough, then from (34) we get by Taylor's expansion

$$\mathcal{L}_{\delta,h} u_0(x_i) = \sum_{l=0}^{\infty} C_l^{\kappa}(h) u_0^{(2l+2)}(x_i), \tag{52}$$

where the coefficients are given by

$$C_l^{\kappa}(h) = 2 \sum_{m=1}^{\kappa+1} \alpha_m \frac{(mh)^{2l+2}}{(2l+2)!}. \tag{53}$$

Lemma 4. *Given the coefficients defined by (53), we have*

$$C_0^{\kappa}(h) = C_\delta, \tag{54}$$

and for $l \geq 1$,

$$C_l^{\kappa}(h) \leq C C_\delta \frac{\max\{\delta, h\}^{2l}}{(2l+2)!} \tag{55}$$

as $h \to 0$.

Proof. By choosing $u_0(z) = z^2$ and setting $x = 0$ in (4), we get

$$\mathcal{L}_\delta u_0(0) = 2 \int_0^{\delta} \gamma_\delta(z) z^2 \, dz = 2C_\delta.$$

On the other hand, by setting $x_i = 0$ in (30), we get

$$\mathcal{L}_{\delta,h} u_0(0) = \sum_{m=-(\kappa+1)}^{\kappa+1} \alpha_m (mh)^2 = 2 \sum_{m=1}^{\kappa+1} \alpha_m (mh)^2 = 2C_0^{\kappa}(h).$$

We know that the quadrature rule (30) is exact for arbitrary quadratic functions, and thus we can obtain (54). (55) can be obtained by noting the fact that $(\kappa+1)h \leq C\delta$ for some $1 < C \leq 2$ if $\kappa > 0$. The proof is completed. □

From (55), we have

$$\sum_{l=1}^{\infty} C_l^{\kappa}(h) u_0^{(2l+2)}(x_i) = O(\max\{\delta, h\}^2), \quad \text{as } \delta, h \to 0$$

for $i \in \mathcal{N}_{in}$. Then from (52) and (54) we see that

$$\mathcal{L}_{\delta,h} u_0(x_i) = C_0^{\kappa}(h) u_0''(x_i) + O(\max\{\delta, h\}^2) \to C_0 u_0''(x_i), \quad \text{as } \delta, h \to 0.$$

Thus we have

$$\|\mathcal{L}_{\delta,h} u_0 - C_0 u_0''\|_\infty \leq C \max\{\delta, h\}^2. \tag{56}$$

Theorem 6. *The collocation scheme (37) for solving the nonlocal diffusion problem (2) is asymptotically compatible, and it holds that*

$$\|u_{\delta,h} - u_0\|_\infty \leq C \frac{1 + 4\delta(1+\delta)}{8C_\delta} \max\{\delta, h\}^2. \tag{57}$$

Proof. From (51) and (37) we have

$$\|u_{\delta,h} - u_0\|_\infty = \|(-\mathcal{L}_{\delta,h})^{-1}(f + \mathcal{L}_{\delta,h} u_0)\|_\infty \leq \|(-\mathcal{L}_{\delta,h})^{-1}\|_\infty \|\mathcal{L}_{\delta,h} u_0 - C_0 u_0''\|_\infty.$$

Connecting it with Proposition 3 and the inequality (56), we easily obtain (57). □

We remark that Theorem 6 also implies the convergence speed of the asymptotic compatibility of the proposed collocation scheme depends on the ratio of δ and h, but is independent of the kernel chosen for the nonlocal diffusion operator.

Table 1
Convergence results of $\|e_h\|_\infty = \|u_{\delta,h} - u_\delta\|_\infty$ with fixed $\delta = 0.1$.

s		$h=\frac{1}{64}$	$h=\frac{1}{128}$	$h=\frac{1}{256}$	$h=\frac{1}{512}$	$h=\frac{1}{1024}$	$h=\frac{1}{2048}$
$-\frac{1}{2}$	$\|e_h\|_\infty$	1.343e-5	1.891e-6	2.423e-7	3.071e-8	3.885e-9	4.892e-10
	ratio		2.828	2.964	2.980	2.983	2.989
0	$\|e_h\|_\infty$	1.554e-5	2.253e-6	2.999e-7	3.889e-8	4.976e-9	6.305e-10
	ratio		2.786	2.910	2.947	2.966	2.980
$\frac{1}{4}$	$\|e_h\|_\infty$	1.763e-5	2.672e-6	3.738e-7	5.049e-8	6.665e-9	8.653e-10
	ratio		2.722	2.837	2.888	2.921	2.945
$\frac{1}{2}$	$\|e_h\|_\infty$	2.170e-5	3.582e-6	5.546e-7	8.289e-8	1.207e-8	1.723e-9
	ratio		2.599	2.691	2.742	2.780	2.809
$\frac{3}{4}$	$\|e_h\|_\infty$	3.155e-5	6.097e-6	1.141e-6	2.098e-7	3.810e-8	6.857e-9
	ratio		2.372	2.417	2.444	2.461	2.474

6. Numerical experiments

We now present some numerical experiments to demonstrate the performance of the proposed collocation scheme (37) for solving the nonlocal diffusion problem (2) and verify the error estimates proved in the preceding section. For all examples, we take the commonly used nonlocal kernel defined in (5) with $s = -\frac{1}{2}, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ as the kernel for the nonlocal diffusion operator.

6.1. Tests with fixed horizon

We choose the exact solution of the nonlocal diffusion problem (2) as

$$u(x) = x^2(1 - x^2).$$

The right-hand side then can be found to be

$$f(x) = 12x^2 - 2 + \frac{2s-2}{s-2}\delta^2.$$

We also fixed $\delta = 0.1$. Table 1 reports the numerical errors and convergence rates, from which we see that the L_∞ errors can achieve approximately $O(h^3)$ if $s < 0.5$, $O(h^3 \ln h)$ if $s = 0.5$ and $O(h^{4-2s})$ if $s > 0.5$, which agree very well with the estimates (49) in Corollary 1.

6.2. Tests for asymptotic compatibility

We now choose the exact solution of the local diffusion problem (51) as

$$u_0(x) = x^2(1 - x^2)$$

and then we have its right-hand side $f := -u_0'' = 12x^2 - 2$ since $C_0 = 1$ for the kernel (5). We investigate the convergence behavior of $u_{\delta,h}$ to u_0 as $\delta, h \to 0$ in three cases: (1) $\delta = h^2$; (2) $\delta = 3h$ with a fixed integer $r > 1$; (3) $\delta = \sqrt{h}$. These cases have been used to investigate the asymptotic compatibility of some finite element schemes in [27].

Table 2 reports the numerical errors and convergence rates for the case of $\delta = h^2$, where δ decreases faster than h while they both go to zero. We can see that the convergence rates are exactly second order for all values of s of the kernel (5) which matches perfectly with the estimate (57) in Theorem 6 by noticing $\max\{\delta, h\}^2 = h^2$ in this case.

Table 3 reports the results for the case of $\delta = 3h$, where both δ and h tend to zero at a fixed proportion. We find that the L_∞ errors still achieve almost perfectly $O(h^2)$ for all s, which again agrees very well with the estimate (57) in Theorem 6 by noticing $\max\{\delta, h\}^2 = 9h^2$ in this case.

Table 4 reports the results for the case of $\delta = \sqrt{h}$, where δ decreases slower than h while they both go to zero. The convergence rates now drop to first order, which again agrees very well with the estimate (57) in Theorem 6 because in this case $\max\{\delta, h\}^2 = h$.

7. Conclusions

In this paper we develop a collocation scheme for solving the nonlocal diffusion problem with general kernels and analyze its convergence behaviors. We rigorously prove that the proposed scheme is second order accurate for the nonlocal

Table 2
Convergence results of $\|e_h\|_\infty = \|u_{\delta,h} - u_0\|_\infty$ for $\delta = h^2$.

s		$h=\frac{1}{64}$	$h=\frac{1}{128}$	$h=\frac{1}{256}$	$h=\frac{1}{512}$	$h=\frac{1}{1024}$	$h=\frac{1}{2048}$
$-\frac{1}{2}$	$\|e_h\|_\infty$	6.104e-5	1.526e-5	3.815e-6	9.537e-7	2.384e-7	5.961e-8
	ratio		2.000	2.000	2.000	2.000	2.000
0	$\|e_h\|_\infty$	6.104e-5	1.526e-5	3.815e-6	9.537e-7	2.384e-7	5.961e-8
	ratio		2.000	2.000	2.000	2.000	2.000
$\frac{1}{4}$	$\|e_h\|_\infty$	6.104e-5	1.526e-5	3.815e-6	9.537e-7	2.384e-7	5.961e-8
	ratio		2.000	2.000	2.000	2.000	2.000
$\frac{1}{2}$	$\|e_h\|_\infty$	6.104e-5	1.526e-5	3.815e-6	9.537e-7	2.384e-7	5.961e-8
	ratio		2.000	2.000	2.000	2.000	2.000
$\frac{3}{4}$	$\|e_h\|_\infty$	6.104e-5	1.526e-5	3.815e-6	9.537e-7	2.384e-7	5.961e-8
	ratio		2.000	2.000	2.000	2.000	2.000

Table 3
Convergence results of $\|e_h\|_\infty = \|u_{\delta,h} - u_0\|_\infty$ for $\delta = 3h$.

s		$h=\frac{1}{64}$	$h=\frac{1}{128}$	$h=\frac{1}{256}$	$h=\frac{1}{512}$	$h=\frac{1}{1024}$	$h=\frac{1}{2048}$
$-\frac{1}{2}$	$\|e_h\|_\infty$	3.629e-4	8.944e-5	2.220e-5	5.530e-6	1.380e-6	3.447e-7
	ratio		2.016	2.008	2.004	2.002	2.001
0	$\|e_h\|_\infty$	3.064e-4	7.575e-5	1.883e-5	4.696e-6	1.172e-6	2.929e-7
	ratio		2.016	2.008	2.004	2.002	2.001
$\frac{1}{4}$	$\|e_h\|_\infty$	2.671e-4	6.618e-5	1.647e-5	4.109e-6	1.026e-6	2.564e-7
	ratio		2.013	2.006	2.003	2.002	2.001
$\frac{1}{2}$	$\|e_h\|_\infty$	2.166e-4	5.381e-5	1.341e-5	3.347e-6	8.362e-7	2.090e-7
	ratio		2.009	2.005	2.002	2.001	2.001
$\frac{3}{4}$	$\|e_h\|_\infty$	1.503e-4	3.745e-5	9.345e-6	2.334e-6	5.833e-7	1.458e-7
	ratio		2.005	2.002	2.001	2.001	2.000

Table 4
Convergence results of $\|e_h\|_\infty = \|u_{\delta,h} - u_0\|_\infty$ for $\delta = \sqrt{h}$.

s		$h=\frac{1}{64}$	$h=\frac{1}{128}$	$h=\frac{1}{256}$	$h=\frac{1}{512}$	$h=\frac{1}{1024}$	$h=\frac{1}{2048}$
$-\frac{1}{2}$	$\|e_h\|_\infty$	2.650e-3	1.284e-3	6.266e-4	3.076e-4	1.517e-4	7.512e-5
	ratio		1.046	1.035	1.026	1.020	1.014
0	$\|e_h\|_\infty$	2.157e-3	1.051e-3	5.151e-4	2.538e-4	1.255e-4	6.228e-5
	ratio		1.038	1.028	1.021	1.016	1.011
$\frac{1}{4}$	$\|e_h\|_\infty$	1.822e-3	8.896e-3	4.374e-4	2.160e-4	1.071e-4	5.318e-5
	ratio		1.034	1.024	1.018	1.013	1.009
$\frac{1}{2}$	$\|e_h\|_\infty$	1.395e-3	6.827e-4	3.365e-4	1.666e-4	8.276e-5	4.118e-5
	ratio		1.031	1.020	1.014	1.010	1.007
$\frac{3}{4}$	$\|e_h\|_\infty$	8.347e-4	4.060e-4	2.000e-4	9.917e-5	4.933e-5	2.459e-5
	ratio		1.040	1.021	1.012	1.007	1.005

problem with fixed horizon, and also it achieves even higher order accuracy for the currently commonly used kernels. Furthermore we show that the scheme is asymptotically compatible with an explicit estimate being derived for the compatibility error. Since only the one-dimensional problem is investigated in this paper, our future work will include its extensions to higher-dimensional problems, and also its generalizations to other nonlocal problems such as the diffusion–convection and the transport ones.

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