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The adaptive composite trapezoidal rule for Hadamard finite-part integrals on an interval

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ABSTRACT

In this article, we discuss an adaptive strategy of implementing trapezoidal rule for evaluating Hadamard finite-part integrals with kernels having different singularity. The purpose is to demonstrate cost savings and fast convergence rates engendered through adaptivity for the computation of finite-part integrals. The error indicators obtained from the *a posteriori* error estimate are used for mesh refinement. Numerical experiments demonstrate that the *a posteriori* error estimate is efficient, and there is no reliability-efficient gap.

1. Introduction

Consider the finite-part integral (see, e.g., [1–3])

$$\mathcal{I}u(y;s) = \ne\!\!\!\!\!\!\int_a^b \frac{u(x)}{|x-y|^{1+2s}}dx, \quad s \in (0,1) \tag{1}$$

with some arbitrary, but fixed $y \in (a,b)$, where $\ne\!\!\!\!\!\int$ denotes an integral in the Hadamard finite-part sense:

$$\ne\!\!\!\!\!\!\int_a^b \frac{u(x)}{|x-y|^{1+2s}}\,dx = \lim_{\epsilon\to 0}\left(\int_{\Omega_\epsilon} \frac{u(x)}{|x-y|^{1+2s}}\,dx - \frac{u(y)}{s\epsilon^{2s}}\right), \tag{2}$$

where $\Omega_\epsilon = (a,b)\setminus(y-\epsilon, y+\epsilon)$. The function $u(x)$ is said to be finite-part integrable with respect to the weight $|x-y|^{-1-2s}$ if the limit on the right hand side of (2) exists.

Integrals of this kind appear in many practical problems related to aerodynamics, wave propagation or fluid mechanics, mostly with relation to boundary element methods(BEMs) and finite-part integral equations [2]. Numerous work has

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been devoted in developing the efficient numerical evaluation method, such as Gaussian method [4,5], Newton–Cotes method [6,1,7–9], and some other methods [10–12]. Amongst them, Newton–Cotes rule is a popular one due to its ease of implementation and flexibility of mesh.

Error analysis of Newton–Cotes rule for Riemann integrals has been well done. The accuracy of Newton–Cotes rule with kth order piecewise polynomial interpolant for the usual Riemann integrals is $O(h^{k+1})$ for odd k and $O(h^{k+2})$ for even k. However, the rule is less accurate for finite-part integral (1) due to the hypersingularity of the kernel. For example, the correspondent result for finite-part integral with first-order singularity ($s = 0$) [13,14] and second-order singularity ($s = \frac{1}{2}$) [1,7,3] is only $O(h^k)$. The superconvergence of composite Newton–Cotes rule for finite-part integral (1) with second-order singularity was investigated in [6,15,3,16], where the higher-order accuracy can be reached on the condition that the singular point coincides with some *a priori* known point. Nevertheless, adaptivity, which is the topic here, is not covered in the above references. The key points in the design of adaptive quadrature rules are, first of all, to keep the number of function evaluations low, and secondly, to divide the domain of integration in such a way that the features of the integrand function are appropriately and effectively accounted for.

We analyze an h-adaptive Newton–Cotes rule for Hadamard finite-part integrals of the form

$$\text{Solve} \rightarrow \text{Estimate} \rightarrow \text{Mark} \rightarrow \text{Refine}; \tag{3}$$

see Section 4 for a precise statement. In the context of the finite element method on shape-regular meshes, h-adaptive algorithms of this type (AFEM) have been analyzed in the last 20 years and are by now fairly well understood [17–22]. The situation is considerably less developed for Newton–Cotes rule for hypersingular integrals [23]. While several *a posteriori* error estimators for Riemann integrals are available in the literature (see [24–26] and the references therein), and numerous numerical studies indicate superiority of adaptive algorithms, an h-adaptive Newton–Cotes rule for Hadamard finite-part integrals still appears to be missing. Such an analysis is the main topic of the present paper.

In this paper, we construct the first-order composite Newton–Cotes rule (trapezoidal rule) for Hadamard finite-part integrals of the form (1), establish *a posteriori* error estimators of residual-type for different singular integral kernels, and design an h-adaptive algorithm based on these estimators. Finally, by means of a series of numerical experiments, we demonstrate that the proposed adaptive quadrature is capable of generating highly accurate approximations at a very low computational cost.

The rest of the paper is organized as follows. Section 2 introduces the precise notation of general (composite) trapezoidal rule for Hadamard finite-part integrals (1). Section 3 presents *a priori* and *a posteriori* error estimates analysis. Section 4 gives details of adaptive algorithms of the trapezoidal rule. Some numerical experiments demonstrate the efficiency of the *a posteriori* error estimates in Section 5.

2. Construction of the composite trapezoidal rule for (1)

As mentioned before, many researchers has made a lot of contributions to composite Newton–Cotes rule for finite-part integrals (see [6,1,7,8]), but all of these works are limited to the situation $s = \frac{1}{2}$. Until recently, a nodal-type trapezoidal rule is developed for (1) with $s \in [0, 1)$ in [9], where the singular point y is always chosen to be located at certain nodal point. In this section, we will derive the (composite) trapezoidal rule for (1) in the case that y is always located in a certain element.

Let

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

be the partition of $[a, b]$ with $h_i = x_{i+1} - x_i$ being the length of the element $e_i = (x_i, x_{i+1})$, $i = 0, 1, \ldots, n-1$. Denote $h = \max_{0 \le i \le n-1} h_i$ the size of the partition. The notation $A \lesssim B$ abbreviates $A \le C \cdot B$ with some generic constant $0 \le C < \infty$, which does not depend on h. We also assume that $y \in e_m$ for certain m satisfying $0 \le m \le n-1$.

Denote the piecewise linear Lagrange interpolant of $u(x)$ by

$$\pi_h u(x) = \sum_{i=0}^{n} u(x_i)\varphi_i(x), \tag{4}$$

where $\varphi_i(x)$ is the piecewise linear hat function, i.e.,

$$\varphi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{h_{i-1}}, & x \in [x_{i-1}, x_i], \\ \dfrac{x_{i+1} - x}{h_i}, & x \in [x_i, x_{i+1}], \\ 0, & \text{otherwise}, \end{cases} \qquad i = 1, \ldots, n-1,$$

and

$$\varphi_0(x) = \begin{cases} \dfrac{x_1 - x}{h_0}, & x \in [x_0, x_1], \\ 0, & \text{otherwise}, \end{cases} \qquad \varphi_n(x) = \begin{cases} \dfrac{x - x_{n-1}}{h_{n-1}}, & x \in [x_{n-1}, x_n], \\ 0, & \text{otherwise}. \end{cases}$$

Replacing $u(x)$ in (1) by $\pi_h u(x)$ leads to the (composite) trapezoidal rule

$$\mathcal{Q}_h u(y;s) = \,=\!\!\!\!\!\!\int_a^b \frac{\pi_h u(x)}{|x-y|^{1+2s}}dx := \sum_{i=0}^{n} \alpha_i^{(s)}(y)u(x_i), \tag{5}$$

where

$$\alpha_i^{(s)}(y) = \begin{cases} =\!\!\!\!\!\!\int_a^b \dfrac{\varphi_i(x)}{|x-y|^{1+2s}}dx, & i=m,\\ \displaystyle\int_a^b \dfrac{\varphi_i(x)}{|x-y|^{1+2s}}dx, & \text{otherwise}. \end{cases}$$

Define

$$\omega_{ij}^{(s)}(y) = \int_{x_i}^{x_{i+1}} \frac{(x-y)^j}{|x-y|^{1+2s}}\,dx, \quad j=0,1,\ldots. \tag{6}$$

It should be noted that $\omega_{mj}^{(s)}(y)$ becomes a finite-part integral if $s < \frac{1}{2}$ and $j=0$ or $s \geq \frac{1}{2}$ and $j=0,1$. By the definition of (2) and through direct calculations, we get

$$\omega_{ij}^{(s)}(y) = \frac{1}{j-2s} \begin{cases} (x_{m+1}-y)^{j-2s} - (-1)^{j+1}(y-x_m)^{j-2s}, & i=m,\\ (-1)^{j+1}\left[(y-x_{i+1})^{j-2s} - (y-x_i)^{j-2s}\right], & i<m,\\ (x_{i+1}-y)^{j-2s} - (x_i-y)^{j-2s}, & i>m \end{cases} \tag{7}$$

for $s \neq \frac{1}{2}$. For the case that $s=\frac{1}{2}$, $\omega_{ij}^{(\frac{1}{2})}(y)$ is the same as (7) except that

$$\omega_{i1}^{\left(\frac{1}{2}\right)}(y) = \ln|(y-x_{i+1})\,/\,(y-x_i)|\,.$$

Once having the explicit expressions of $\omega_{ij}^{(s)}(y)$, the coefficients $\alpha_i^{(s)}(y)$ can be reformulated as follows:

$$\alpha_i^{(s)}(y) = \begin{cases} -h_0^{-1}\omega_{01}^{(s)}(y) + \omega_{00}^{(s)}(y)\dfrac{x_1-y}{h_0}, & i=0,\\ h_{i-1}^{-1}\omega_{i-1,1}^{(s)}(y) + \omega_{i-1,0}^{(s)}(y)\dfrac{y-x_{i-1}}{h_{i-1}} - h_i^{-1}\omega_{i1}^{(s)}(y) + \omega_{i0}^{(s)}(y)\dfrac{x_{i+1}-y}{h_i}, & i=1,\ldots,n-1,\\ h_{n-1}^{-1}\omega_{n-1,1}^{(s)}(y) + \omega_{n-1,0}^{(s)}(y)\dfrac{y-x_{n-1}}{h_i}, & i=n. \end{cases}$$

If $s=\frac{1}{2}$, (5) is no other than the rule discussed in [8].

3. Error estimate

For finite-part integral (1) with second-order singularity, the error estimate of trapezoidal rule (5) has been provided in [1,8] on quasi-uniform meshes. However, our purpose is to develop an adaptive strategy for implementing (5), and thus a key issue for us is to obtain *a priori* and *a posteriori* error estimates of (5) on arbitrary meshes, which constitutes the main topic of this section.

3.1. A priori error estimate of (5)

Before the statement of *a priori* error estimate of (5), we first consider a classic trapezoidal rule for the following integral

$$\int_0^X f(x)\,dx, \qquad f(x) = x^{-2s+1}, \quad s \in (0,1). \tag{8}$$

To describe this rule, we need to introduce a partition of $[0, X]$, which says

$$\Pi_h : 0 = \bar{x}_0 < \bar{x}_1 < \cdots < \bar{x}_{N-1} < \bar{x}_N = X.$$

Let $\bar{h}_i = \bar{x}_{i+1} - \bar{x}_i$, $i=0,1,\ldots,N-1$ be the length of each element and $\bar{h} = \max_i \bar{h}_i$ be the size of the partition. Note that (8) is at most weakly singular, by using the method of "avoiding the singularity", the classic trapezoidal rule for evaluating (8) can be formulated as

$$T(f) := \sum_{i=1}^{N-1} \frac{f(\bar{x}_i) + f(\bar{x}_{i+1})}{2}\bar{h}_i. \tag{9}$$

Lemma 1. *Under the partition Π_h, for the classic trapezoidal rule (9), we have*

$$|E(f,T)| = \left|\int_0^X f(x)\,dx - T(f)\right| \lesssim \bar{h}_0^{-2s+2} + \bar{h}^2/\bar{h}_0^{2s} + \bar{h}^2. \tag{10}$$

Proof. Using integration by parts twice, we have

$$\int_{\bar{x}_i}^{\bar{x}_{i+1}} f(x)\,dx = \frac{f(\bar{x}_i)+f(\bar{x}_{i+1})}{2}\bar{h}_i + \frac{1}{2}\int_{\bar{x}_i}^{\bar{x}_{i+1}} (x-\bar{x}_i)(x-\bar{x}_{i+1})f''(x)\,dx. \tag{11}$$

Further direct calculations show that

$$\begin{aligned} E(f,T) &= \int_{\bar{x}_0}^{\bar{x}_1} f(x)\,dx + \sum_{i=1}^{N-1}\int_{\bar{x}_i}^{\bar{x}_{i+1}} f(x)\,dx - T(f) \\ &= \frac{\bar{h}_0^{-2s+2}}{-2s+2} + \frac{1}{2}\sum_{i=1}^{N-1}\int_{\bar{x}_i}^{\bar{x}_{i+1}} (x-\bar{x}_i)(x-\bar{x}_{i+1})f''(x)\,dx, \end{aligned} \tag{12}$$

which implies that

$$\begin{aligned} |E(f,T)| &\le \frac{\bar{h}_0^{-2s+2}}{-2s+2} + \frac{\bar{h}^2}{2}\int_{\bar{h}_0}^{X} |f''(x)|\,dx \\ &= \frac{\bar{h}_0^{-2s+2}}{-2s+2} + \frac{|-2s+1|\bar{h}^2}{2}\left(\bar{h}_0^{-2s} - X^{-2s}\right) \\ &\lesssim \bar{h}_0^{-2s+2} + \bar{h}^2/\bar{h}_0^{2s} + \bar{h}^2. \end{aligned}$$

This concludes the proof. □

Remark 2. Lemma 1 shows the error estimate of trapezoidal rule (9) for arbitrary meshes. If the uniform mesh is adopted, we can simplify (10) as $|E(f,T)| \lesssim \bar{h}^{-2s+2}$.

Define $\mathcal{R}_i(x) := u(x) - \pi_h u(x),\ x \in e_i$ and let

$$\mathcal{E}_i = \begin{cases} \displaystyle ⨎_{x_i}^{x_{i+1}} \frac{\mathcal{R}_i(x)}{|x-y|^{1+2s}}\,dx, & i = m, \\ \displaystyle \int_{x_i}^{x_{i+1}} \frac{\mathcal{R}_i(x)}{|x-y|^{1+2s}}\,dx, & \text{otherwise} \end{cases} \tag{13}$$

denote the error of trapezoidal rule on element e_i. The remaining parts of this section are devoted to *a priori* error estimate of the (composite) trapezoidal rule (5).

Theorem 3 (*A Priori Error Estimate*)**.** *Assume that $u(x) \in C^2[a,b]$ and the singular point y is located at the mid-point of certain element e_m. The error of the (composite) trapezoidal rule (5) satisfies*

$$|\mathcal{I}u(y;s) - \mathcal{Q}_h u(y;s)| \lesssim \|u''\|_\infty \begin{cases} h + h_m|\ln h_m| + h^2/h_m, & s = 0.5, \\ h^2 + h_m^{2-2s} + h^2/h_m^{2s}, & \textit{otherwise}. \end{cases} \tag{14}$$

Proof. For any $u(x) \in C^2[a,b]$, the error of (composite) trapezoidal rule (5) can be rewritten as

$$\mathcal{I}u(y;s) - \mathcal{Q}_h u(y;s) := \epsilon_1 + \epsilon_2 + \epsilon_3,$$

where

$$\epsilon_1 = \mathcal{E}_m, \qquad \epsilon_2 = \sum_{i=0}^{m-1} \mathcal{E}_i, \qquad \epsilon_3 = \sum_{i=m+1}^{n-1} \mathcal{E}_i. \tag{15}$$

Now, we estimate ϵ_i term by term. For ϵ_1, the definition of $\omega_{mj}^{(s)}(y)(j=0,1)$ implies that

$$\epsilon_1 = \omega_{m0}^{(s)}(y)\mathcal{R}_m(y) + \omega_{m1}^{(s)}(y)\mathcal{R}'_m(y) + \int_{x_m}^{x_{m+1}} \frac{\mathcal{R}_m(x) - \mathcal{R}_m(y) - \mathcal{R}'_m(y)(x-y)}{|x-y|^{1+2s}}\,dx. \tag{16}$$

The interpolant theorem shows

$$\left|\omega_{m0}^{(s)}(y)\mathcal{R}_m(y)\right| \lesssim \|u''\|_{\infty,e_m} h_m^2 |\omega_{m0}^{(s)}(y)|, \tag{17}$$

$$\left|\omega_{m1}^{(s)}(y)\mathcal{R}_m'(y)\right| \lesssim \|u''\|_{\infty,e_m} h_m |\omega_{m1}^{(s)}(y)|. \tag{18}$$

Note that the integral on the right-hand side of (16) has been reduced to a Riemann integral since its integrand function only possesses a removable discontinuity at $x = y$. Thus,

$$\begin{aligned}\left|\int_{x_m}^{x_{m+1}} \frac{\mathcal{R}_m(x) - \mathcal{R}_m(y) - \mathcal{R}_m'(x)(x-y)}{|x-y|^{1+2s}}\, dx\right| &= \frac{1}{2}\left|\int_{x_m}^{x_{m+1}} \frac{\mathcal{R}_m''(x_m + \theta_m h_m)(x-y)^2}{|x-y|^{1+2s}}\, dx\right| \\ &\lesssim \|u''\|_{\infty,e_m} |\omega_{m2}^{(s)}(y)|, \end{aligned} \tag{19}$$

where $\theta_m \in (0, 1)$. The sum of (17)–(19) yields

$$\begin{aligned}|\epsilon_1| &\lesssim \|u''\|_{\infty,e_m} \left(h_m^2|\omega_{m0}^{(s)}(y)| + h_m|\omega_{m1}^{(s)}(y)| + |\omega_{m2}^{(s)}(y)|\right) \\ &\lesssim \|u''\|_{\infty,e_m} \begin{cases} h_m|\ln h_m|, & s = \frac{1}{2}, \\ h_m^{2-2s}, & \text{otherwise}. \end{cases}\end{aligned} \tag{20}$$

In the following, we will mainly focus on the estimate of ϵ_3, and the estimate of ϵ_2 can be done in a similar way. The Lagrange interpolation theory shows

$$\mathcal{R}_i(x) = \frac{1}{2}u''(x_i + \theta_i h_i)(x - x_i)(x - x_{i+1}), \quad x \in [x_i, x_{i+1}],\ \theta_i \in (0, 1),$$

this implies

$$\begin{aligned}|\epsilon_3| &\lesssim \|u''\|_\infty \sum_{i=m+1}^{n-1} \int_{x_i}^{x_{i+1}} \frac{(x - x_i)(x_{i+1} - x)}{(x-y)^{1+2s}}\, dx \\ &\lesssim \|u''\|_\infty \left(\int_{h_m/2}^{b-y} t^{-2s+1}\, dt - \frac{1}{2}\sum_{i=m+1}^{n-1}\left[(x_{i+1}-y)^{-2s+1} + (x_i - y)^{-2s+1}\right] h_i\right).\end{aligned}$$

Since the first term in the bracket is actually the integral (8) if we choose $X = b - y$, and the second term is just the classic trapezoidal rule (9) if we choose $N = n - m - 1$ and $\bar{h}_0 = h_m/2$, $\bar{h}_i = h_{m+i}$, $i = 1, 2, \ldots, N - 1$, by Lemma 1, we have

$$|\epsilon_3| \lesssim \|u''\|_\infty \left(h^2 + h_m^{-2s+2} + h^2/h_m^{2s}\right).$$

The combination of above estimates concludes the proof. □

3.2. *A posteriori error estimate of (5)*

This section is devoted to an a posteriori error analysis of the (composite) trapezoidal rule.

Theorem 4. *Assume that $u(x) \in C^2[a, b]$ and the singular point y is located at the mid-point of certain element e_m. Then, the error estimates of the trapezoidal rule (5) at each element satisfy*

$$|\mathcal{E}_m| \lesssim \|u''\|_{\infty,e_m} \begin{cases} h_m|\ln h_m|, & s = \frac{1}{2}, \\ h_m^{2-2s}, & \textit{otherwise} \end{cases} \tag{21}$$

and if $i \neq m$,

$$|\mathcal{E}_i| \lesssim \|u''\|_{\infty,e_i} d_i^{-1-2s} h_m^{-1-2s} h_i^3, \tag{22}$$

where

$$d_i = h_m^{-1} \min\{|y - x_i|, |y - x_{i+1}|\}. \tag{23}$$

Proof. (21) is followed by (20). It remains us to estimate $\mathcal{E}_i$, $i \neq m$. The classic error estimate of linear Lagrangian interpolation shows,

$$\mathcal{R}_i(x) = \frac{u''(x_i + \theta_i h_i)}{2!}(x - x_i)(x - x_{i+1}) \quad \theta_m \in (0, 1).$$

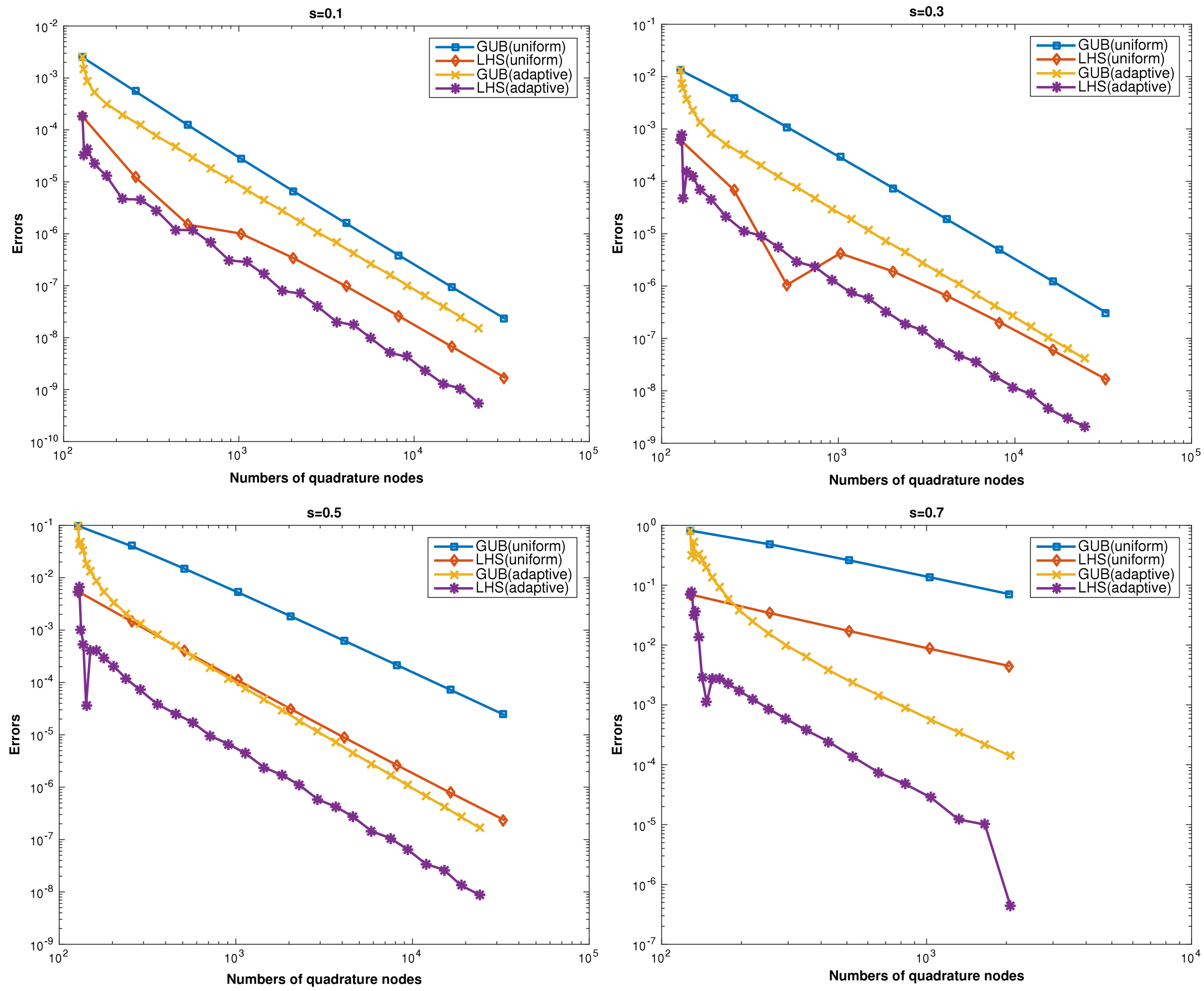

Fig. 1. Errors of trapezoidal rule for $u(x) = x^2(1-x)^2$ and $y = 0.5$.

Hence,

$$
\begin{aligned}
|\mathcal{E}_i| &\lesssim \int_{x_i}^{x_{i+1}} \frac{|\mathcal{R}_i(x)|}{|x-y|^{1+2s}}\,dx \\
&\lesssim \|u''\|_{\infty,e_i} d_i^{-1-2s} h_m^{-1-2s} \int_{x_i}^{x_{i+1}} |(x-x_i)(x-x_{i+1})|\,dx \\
&\lesssim \|u''\|_{\infty,e_i} d_i^{-1-2s} h_m^{-1-2s} h_i^3.
\end{aligned}
$$

This concludes the proof. □

Theorem 5 (*A Posteriori Error Estimate*). *Assume that $u(x) \in C^2[a,b]$ and the singular point y is located at the mid-point of certain element e_m. Then, the error of the trapezoidal rule (5) satisfies*

$$
|\mathcal{I}u(y;s) - \mathcal{Q}_h u(y;s)| \lesssim \sum_{i=0, i\neq m}^{n-1} \|u''\|_{\infty,e_i} d_i^{-1-2s} h_m^{-1-2s} h_i^3 + \|u''\|_{\infty,e_m} \begin{cases} h_m |\ln h_m|, & s = \dfrac{1}{2}, \\ h_m^{2-2s}, & \text{otherwise}. \end{cases} \tag{24}
$$

Proof. The proof can be obtained by taking the sum of errors in (21) and (22). □

For convenience, denote the left-hand side of (24) by LHS and it is actually the errors of the trapezoidal rule (5). Besides, the right-hand side of (24) is often called *guaranteed upper bounds* (GUB).

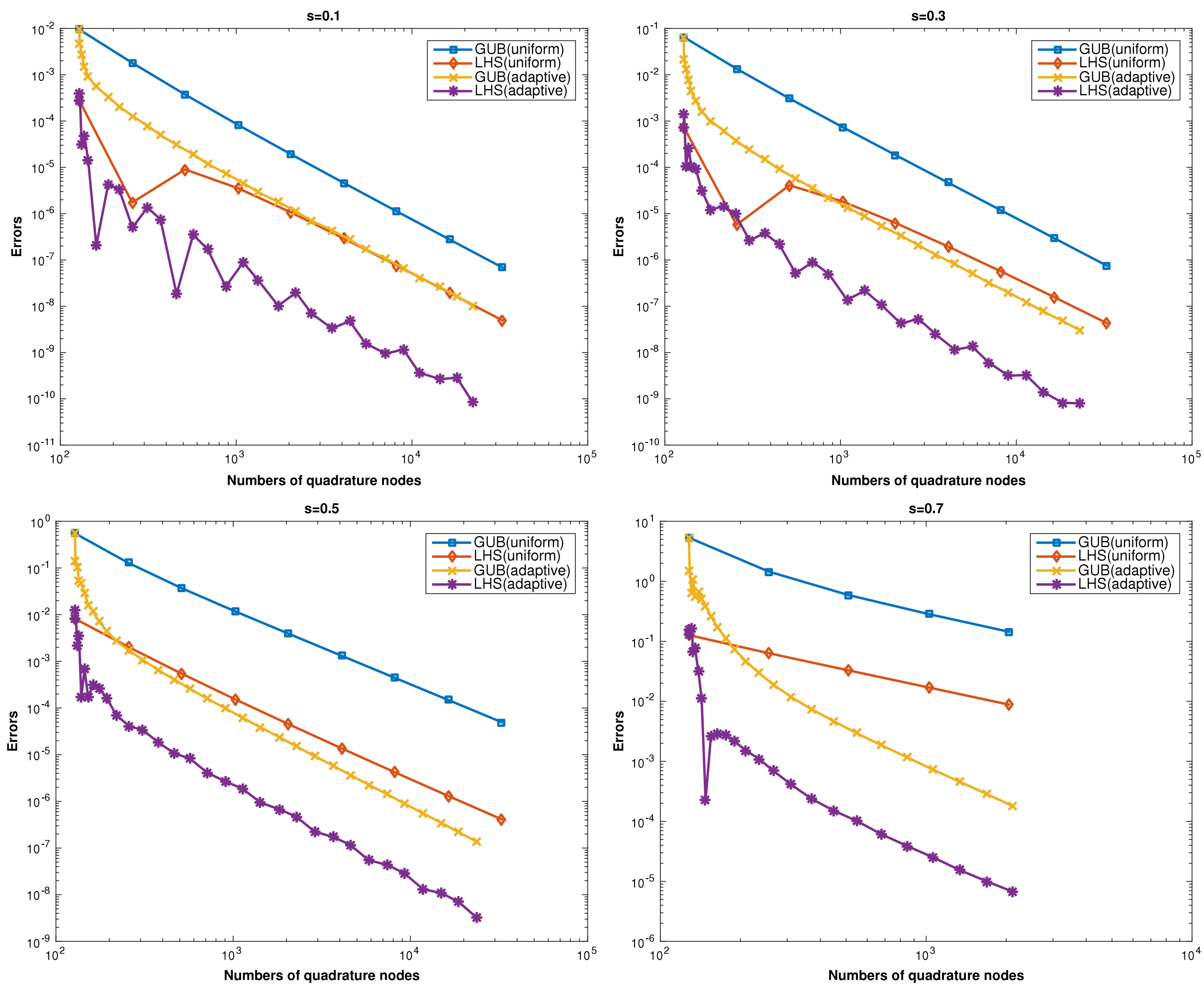

Fig. 2. Errors of trapezoidal rule for $u(x) = x^2(1-x)^2$ and $y = 10^{-5}$.

4. Adaptive algorithms of (5)

This section describes the adaptive mesh-refinement algorithms of trapezoidal rule, which compute a sequence of meshes based on a loop over levels of the steps

$$\text{Solve} \longrightarrow \text{Estimate} \longrightarrow \text{Mark} \longrightarrow \text{Refine}.$$

4.1. Solve

We can always choose an initial coarse regular partition $\mathcal{N}_0$ of interval $I = [a, b]$ such that the singular point y is located at a mid-point of certain subinterval e_m. In this case, we put the level counter $\ell = 0$. For $\ell = 0, 1, 2 \ldots$ (until termination), apply the (composite) trapezoidal rule (5) to evaluate the integral (1).

4.2. Estimate

Compute LHS and GUB. Note that GUB contains $\|u''\|_{\infty,e_i}$, and in practice we hope to avoid the second derivatives of u. A natural approximation is

$$\|u''\|_{\infty,e_i} \approx \left| \frac{u(x_i) - 2u\left(x_{i+\frac{1}{2}}\right) + u(x_{i+1})}{(h_i/2)^2} \right|.$$

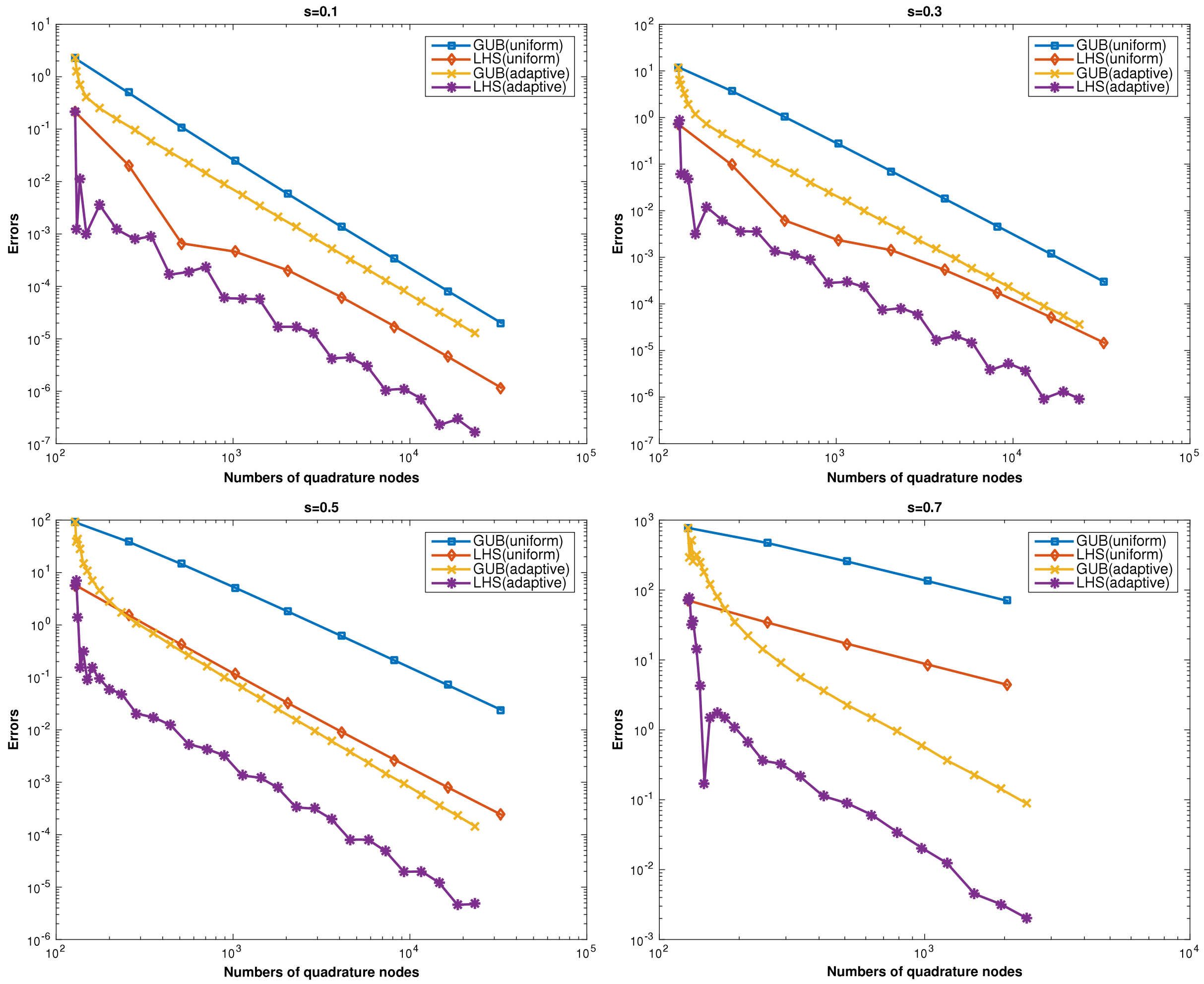

Fig. 3. Errors of trapezoidal rule for $u(x) = \cos(10\pi x)$ and $y = 0.5$.

4.3. Mark

For any subinterval $[x_i, x_{i+1}]$, set

$$\eta_i = \|u''\|_{\infty,e_i} \begin{cases} h_m^{2-2s}, & i = m, \\ d_i^{-1-2s} h_m^{-1-2s} h_i^3, & \text{otherwise} \end{cases} \tag{25}$$

for $s \neq \frac{1}{2}$ and if $s = \frac{1}{2}$ then h_m^{2-2s} in (25) should be replaced by $h_m|\ln h_m|$. Given all those contributions in (25), mark some set $\mathcal{M}$ of intervals in $\mathcal{N} = \{0, 1, \ldots, n-1\}$ of minimal cardinality with the bulk criterion

$$\theta \sum_{j\in\mathcal{N}} \eta_j \leq \sum_{j\in\mathcal{M}} \eta_j, \tag{26}$$

where θ is the adaptivity parameter which satisfies $0 < \theta \leq 1$.

4.4. Refine

The adaptive mesh-refinement algorithm is described as follows: If the element $e = e_m$ is marked, then refine it by trisection. Under this refinement, the singular point y can be guaranteed to be located at the middle point of the subinterval e_m. Otherwise, refine the element e by bisection.

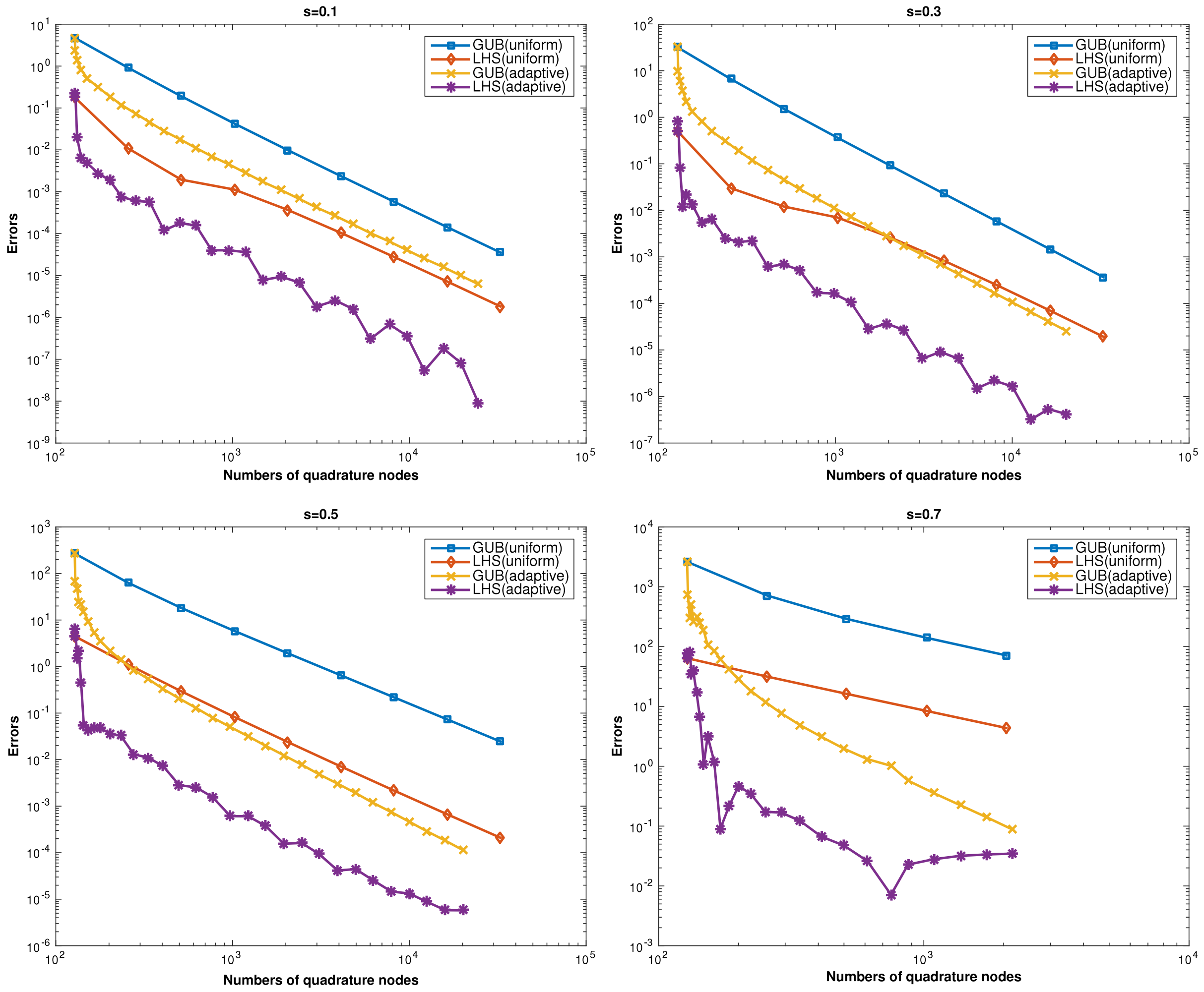

Fig. 4. Errors of trapezoidal rule for $u(x) = \cos(10\pi x)$ and $y = 10^{-5}$.

5. Numerical experiments

This section is devoted to the numerical investigation of the trapezoidal rule (5) for finite-part integral (1). We choose $a = 0, b = 1$ and test two functions: $u(x) = x^2(1-x)^2$ and $u(x) = \cos(10\pi x)$. Both are analytic but the latter is highly oscillating. We test two cases: (1) the singular point y is located at the middle of the interval, i.e., $y = 0.5$; (2) the singular point y is very close to the endpoints, i.e., $y = 10^{-5}$.

We compare the errors on the uniform meshes ($\theta = 1$) and adaptive meshes ($\theta = 0.5$). Numerical results for the first analytic function are demonstrated in Figs. 1 and 2 and the results for the second analytic but highly oscillating function are demonstrated in Figs. 3 and 4. All these results demonstrate that GUB captures the convergence rate of LHS, and there is no reliability-efficient gap, which agrees well with the estimate (24) in Theorem 5. Furthermore, numerical experiments also show that no matter where the singular point is located at, the adaptive strategy of trapezoidal rule shows superiority over the uniform refinement strategy, and meanwhile the severer the singularity s is, this superiority is more obvious.

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