SCIENCE CHINA
Mathematics

• ARTICLES •

August 2013 Vol. 56 No. 8: 1647–1660
doi: 10.1007/s11425-013-4593-1

Extrapolation methods to compute hypersingular integral in boundary element methods

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Received January 29, 2012; accepted June 6, 2012; published online March 29, 2013

Abstract The composite trapezoidal rule for the computation of Hadamard finite-part integrals in boundary element methods with the hypersingular kernel $1/\sin^2(x-s)$ is discussed, and the main part of the asymptotic expansion of error function is obtained. Based on the main part of the asymptotic expansion, a series is constructed to approach the singular point. An extrapolation algorithm is presented and the convergence rate is proved. Some numerical results are also presented to confirm the theoretical results and show the efficiency of the algorithms.

Keywords hypersingular integrals, trapezoidal rule, asymptotic error expansion, extrapolation algorithm

MSC(2010) 35S15, 65M38, 74S15, 30B50

Citation: Li J, Zhang X P, Yu D H. Extrapolation methods to compute hypersingular integral in boundary element methods. Sci China Math, 2013, 56: 1647–1660, doi: 10.1007/s11425-013-4593-1

1 Introduction

Singular integral equations, especially hypersingular integral equations, are usually encountered in boundary element methods [7, 27–29, 31]; and in a range of nonlinear mathematical models [2, 6, 8, 9]; such as acoustics, fluid mechanics, elasticity, fracture mechanics and electromagnetic scattering problems and so on. Together with the physical problems from such areas and the resulting mathematical models, much attention has been paid to the hypersigular integral of the form

$$\neq\!\!\!\!\!\!\int_a^b \frac{f(t)}{(t-s)^2}dt = \lim_{\varepsilon\to 0}\left\{\int_a^{s-\varepsilon}\frac{f(t)}{(t-s)^2}dt + \int_{s+\varepsilon}^b \frac{f(t)}{(t-s)^2}dt - \frac{2f(s)}{\varepsilon}\right\}, \quad s\in(a,b), \tag{1.1}$$

where $\neq\!\!\!\!\!\!\int_a^b$ denotes a Hadamard finite-part integral and s the singular point.

Hypersinguler integral must be considered in Hadamard finite-part sense. Numerous work has been done in developing efficient quadrature formulas in recent years, such as the Gaussian method [9, 10], the transformation method [5, 6] and some other methods [4, 11, 16–18, 24, 26, 30].

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This paper focuses on a less studied hypersingular integral on circle which appears frequently in the formulation of certain classes of boundary value problems in a circular or an elliptic domain. We present the following definition,

$$I(f;s) = \oint_c^{c+2\pi} \frac{f(t)}{\sin^2 \frac{t-s}{2}} dt = \lim_{\varepsilon \to 0} \left\{ \left(\int_c^{s-\varepsilon} + \int_{s+\varepsilon}^{c+2\pi} \right) \frac{f(t)}{\sin^2 \frac{t-s}{2}} dt - 4f(s)\cot\frac{\varepsilon}{2} \right\}, \tag{1.2}$$

where $s \in (c, c+2\pi)$ and the integrals on the right-hand side of (1.2) are the usual Riemann integrals. $f(t)$ is said to be finite-part integrable with the weight $\sin^2 \frac{t-s}{2}$ if limit in (1.2) exists.

Due to the hypersingularity of the integrand, the convergence rate for Hadamard finite-part integrals is only $O(h)$ much lower than its counterpart for Riemann integrals. For this kind of hypersingular integral, Yu [28, 29] firstly generalized the kernel expansion methods to solve the hypersingular integral equations. Authors of [32] discussed the composite trapezoidal rule and its superconvergence phenomenon for such integral. Then in [33], they futher studied the general Newton-Cotes rules and their corresponding superconvergence result. The superconvergent rate $O(h^{k+1})$ were obtained when the singular point coincided with the some special points. Obviously, one can obtain better accuracy if the higher order Newton-Cotes rule is chosen. But there exists a drawback that the Cotes-coefficients of the rule do not possess the simply explicit expression when $k \geqslant 2$, which shows that the trapezoidal rule is the simplest one among the Newton-Cotes rules. Trapezoidal rule can only produce the accuracy $O(h)$ in a general case and the superconvergent rate $O(h^2)$, which seems inadequate in practical computation. A direct way to use this rule more efficiently is the extrapolation method. In this paper, we will mainly discuss the asymptotic error expansion of the trapezoidal rule and the construction of the corresponding extrapolation algorithm to accelerate the convergence rate.

The classical extrapolation method based on polynomial and rational function has been widely studied. As an accelerating convergence technique, the extrapolation method has been applied to many fields in computational mathematics [12, 25]. In [22], based on a generalization of discrete Gronwall inequality, a new quadrature method for solving nonlinear weakly singular Volterra integral equations of the second kind is presented and the asymptotic expansion of the error is proved. Then, in [14], an extrapolation method is presented to compute hypersingular integrals at intervals, and the arbitrary convergence rate can be obtained if the density function $f(t)$ is smooth enough.

Let

$$\psi_{ik}(t) = \begin{cases} \displaystyle\oint_{-1}^{1} \frac{\mathcal{M}_{ik}(\tau,t)}{(k-i)!\sin^2\frac{\tau-t}{2}} d\tau, & |t| < 1, \\ \displaystyle\int_{-1}^{1} \frac{\mathcal{M}_{ik}(\tau,t)}{(k-i)!\sin^2\frac{\tau-t}{2}} d\tau, & |t| > 1, \end{cases} \tag{1.3}$$

where

$$\mathcal{M}_{ik}(\tau,t) = (\tau^2-1)(\tau-t)^{k-i}[(\tau+1)^i - (\tau-1)^i. \tag{1.4}$$

Let $J := (-,\infty) \cup (-1,1) \cup (1,\infty)$ and the operator $W : C(J) \to C(-1,1)$ be defined by

$$T_{ik}(\tau) := \psi_{ik}(\tau) + \sum_{j=1}^{\infty} \psi_{ik}(2j+\tau) + \sum_{j=1}^{\infty} \psi_{ik}(-2j+\tau), \quad k = i, i+1. \tag{1.5}$$

Then, the asymptotic error expansion of the trapezoidal rule for computing (1.2) takes the form of

$$E_n(f) = \sum_{i=1}^{l-1} \frac{h^i}{2^i} f^{(i+1)}(s) a_i(\tau) + O(h^{l-1}), \tag{1.6}$$

where $a_i(\tau)$ are functions independent of h defined as

$$a_i(\tau) = \begin{cases} \displaystyle\sum_{k=i-1}^{i} \frac{(-1)^k}{(k+1)!} T_{ki}(\tau), & i > 1, \\ -\dfrac{1}{2} T_{ii}(\tau), & i = 1, \end{cases} \tag{1.7}$$

and τ is the local coordinate of the singular point.

Based on the main part of the asymptotic expansion (1.6), in order to get higher convergence rate, we suggest an extrapolation algorithm: for a given τ, a series of s_j is selected to approximate the singular point s accompanied by the refinement of the meshes. Moreover, by means of the extrapolation technique, we not only obtain an approximation with higher order accuracy, but also get a posteriori error estimate.

The rest of this paper is organized as follows. In Section 2, after introducing some basic formulas of the trapezoidal rule, we present the main part of the asymptotic error expansion. In Section 3, some lemmas are given and the proof of the main result is completed. In Section 4, extrapolation algorithm and a posteriori asymptotic error estimation to compute Hadamard finite-part integral are obtained. Finally, several numerical examples are provided to validate our analysis.

2 Main result

Let $c = t_0 < t_1 < \cdots < t_{n-1} < t_n = c + 2\pi$ be a uniform partition of the interval $[c, c+2\pi]$ with mesh size $h = 2\pi/n$. We define the piecewise linear interpolation for $f(t)$,

$$f_L(t) = \frac{t - t_{j-1}}{h} f(t_j) + \frac{t_j - t}{h} f(t_{j-1}), \quad t \in [t_{j-1}, t_j], \quad 1 \leqslant j \leqslant n, \tag{2.1}$$

and a linear transformation

$$t = \hat{t}_j(\tau) := (\tau + 1)(t_j - t_{j-1})/2 + t_{j-1}, \quad \tau \in [-1, 1], \tag{2.2}$$

transform the reference element $[-1, 1]$ to the subinterval $[t_{j-1}, t_j]$. Replacing $f(t)$ in (1.2) with $f_L(t)$ gives the composite trapezoidal rule:

$$I_n(f; s) := \neq\!\!\!\!\!\!\int_c^{c+2\pi} \frac{f_L(t)}{\sin^2 \frac{t-s}{2}} dt = \sum_{j=1}^{n} \omega_j(s) f(t_j) = I(f; s) - E_n(f), \tag{2.3}$$

where $\omega_j(s)$ denotes the Cotes coefficient given by

$$\omega_j(s) = \frac{4}{h} \ln \left| \frac{1 - \cos(t_j - s)}{\cos h - \cos(t_j - s)} \right|. \tag{2.4}$$

Before presenting the main results, we firstly define $K_s(t)$ as follows,

$$K_s(t) = \begin{cases} \dfrac{(t-s)^2}{\sin^2 \frac{t-s}{2}}, & t \neq s, \\ 4, & t = s. \end{cases} \tag{2.5}$$

Theorem 2.1. *Assume $f(t) \in C^l[a, b], l \geqslant 2$. For the trapezoidal rule $I_n(f; s)$ defined in (2.3) and $a_i(\tau)$ defined in (1.7), there exists a positive constant C, independent of h and s, such that*

$$E_n(f) = \sum_{i=1}^{l-1} \frac{h^i}{2^i} f^{(i+1)}(s) a_i(\tau) + \mathcal{R}_n(s), \tag{2.6}$$

where $s = t_{m-1} + (1+\tau)h/2,\ m = 1, 2, \ldots, n$, $K_s(t)$ defined by (2.5) and

$$|\mathcal{R}_n(s)| \leqslant C \max_{t \in (c, c+2\pi)} \{K_s(t)\}(|\ln h| + \gamma^{-1}(\tau)) h^{l-1}$$

and

$$\gamma(\tau) = \frac{1 - |\tau|}{2}.$$

3 Proof of the theorem

Before starting our proof, we recall some notation in [14]. They introduced function

$$\mathcal{M}_{ik}^{j}(t,s)=\mathcal{F}_{i}^{j}(t)(t-s)^{k-i},\quad k\geqslant i,\tag{3.1}$$

where

$$\mathcal{F}_{i}^{j}(t)=(t-t_{j-1})(t-t_{j})[(t-t_{j-1})^{i}-(t-t_{j})^{i}].\tag{3.2}$$

By (2.2), we have

$$\mathcal{M}_{ik}^{j}(t,s)=\frac{h^{k+2}}{2^{k+2}}\mathcal{F}_{i}(\tau)(\tau-c_j)^{k-i}=\frac{h^{k+2}}{2^{k+2}}\mathcal{M}_{ik}(\tau,c_j),\tag{3.3}$$

where

$$\mathcal{M}_{ik}(\tau,c_j)=(\tau^2-1)(\tau-c_j)^{k-i}[(\tau+1)^{i}-(\tau-1)^{i}],\tag{3.4}$$

$$c_j=2(s-t_{j-1})/h-1,\tag{3.5}$$

and

$$\mathcal{F}_{i}(\tau)=(\tau^2-1)[(\tau+1)^{i}-(\tau-1)^{i}].\tag{3.6}$$

In the following analysis, C will denote a generic positive constant which may be different at different places and independent of h and s, but which may depend on k and bounds of the derivatives of $f(t)$.

Lemma 3.1. *Let $K_s(t)$ be defined in (2.5). For $t\in(t_{j-1},t_j)$, by linear transformation (2.2), we have*

$$K_s(t)=K_{c_j}(\tau),\quad \tau\in(-1,1),\tag{3.7}$$

where

$$K_{c_j}(\tau)=4+4\sum_{l=1}^{\infty}\frac{(\tau-c_j)^2}{(\tau-c_j-2ln)^2}+4\sum_{l=1}^{\infty}\frac{(\tau-c_j)^2}{(\tau-c_j+2ln)^2}\tag{3.8}$$

and c_j is defined as (3.5).

Proof. By the identity in [1],

$$\frac{\pi^2}{\sin^2\pi t}=\sum_{l=-\infty}^{l=\infty}\frac{1}{(t+l)^2},\tag{3.9}$$

then we get

$$\frac{1}{\sin^2\frac{t-s}{2}}=\frac{4}{(t-s)^2}+4\sum_{l=1}^{\infty}\frac{1}{(t-s-2l\pi)^2}+4\sum_{l=1}^{\infty}\frac{1}{(t-s+2l\pi)^2}\tag{3.10}$$

and

$$\begin{aligned}K_s(t)=\frac{(t-s)^2}{\sin^2\frac{t-s}{2}}&=4+4\sum_{l=1}^{\infty}\frac{(\tau-c_j)^2}{(\tau-c_j-4l\pi/h)^2}+4\sum_{l=1}^{\infty}\frac{(\tau-c_j)^2}{(\tau-c_j+4l\pi/h)^2}\\&=4+4\sum_{l=1}^{\infty}\frac{(\tau-c_j)^2}{(\tau-c_j-2ln)^2}+4\sum_{l=1}^{\infty}\frac{(\tau-c_j)^2}{(\tau-c_j+2ln)^2}\\&=K_{c_j}(\tau),\end{aligned}$$

which completes the proof. □

Lemma 3.2. *Assume $s\in(t_{m-1},t_m)$, for some m, c_j defined as (3.5) and $k=i$. Then, we have*

$$\psi_{ik}(c_j)=\begin{cases}-\dfrac{2^k}{h^{k+1}}\displaystyle⨍_{t_{j-1}}^{t_j}\frac{\mathcal{M}_{ik}^{j}(t,s)}{\sin^2\frac{t-s}{2}}dt, & j=m,\\[2ex] -\dfrac{2^k}{h^{k+1}}\displaystyle\int_{t_{j-1}}^{t_j}\frac{\mathcal{M}_{ik}^{j}(t,s)}{\sin^2\frac{t-s}{2}}dt, & j\neq m.\end{cases}\tag{3.11}$$

Proof. By the definition of (1.2), we have

$$\begin{aligned}
\text{⨎}_{t_{j-1}}^{t_j} \frac{\mathcal{M}_{ik}^j(t,s)}{\sin^2\frac{t-s}{2}}dt &= \text{⨎}_{t_{j-1}}^{t_j} \frac{F_i^j(t)K_s(t)}{(t-s)^2}dt \\
&= \lim_{\varepsilon\to 0}\left\{\int_{t_{j-1}}^{s-\varepsilon} \frac{F_i^j(t)K_s(t)}{(t-s)^2}dt + \int_{s+\varepsilon}^{t_j} \frac{F_i^j(t)K_s(t)}{(t-s)^2}dt - \frac{2F_i^j(s)K_s(s)}{\varepsilon}\right\} \\
&= \frac{h^{k+1}}{2^{k+1}}\lim_{\varepsilon\to 0}\left\{\left(\int_{-1}^{c_m-\frac{2\varepsilon}{h}} + \int_{c_m+\frac{2\varepsilon}{h}}^{1}\right)\frac{F_i(\tau)K_s(\tau)}{(\tau-c_m)^2}d\tau - \frac{hF_i(c_m)K_s(c_m)}{\varepsilon}\right\} \\
&= \frac{h^{k+1}}{2^{k+1}} \text{⨎}_{-1}^{1} \frac{F_i(\tau)K_s(\tau)}{(\tau-c_m)^2}d\tau \\
&= -\frac{h^{k+1}}{2^k}\psi_{ii}(c_m). \qquad (3.12)
\end{aligned}$$

The first identity in (3.11) is then verified. The second identity can be obtained by applying the approach to the correspondent Riemann integral. The proof is completed. □

Lemma 3.3. *Under the assumption of Lemma* 3.2 *and* $k=i+1$. *Then, we have*

$$\psi_{ik}(c_j) = \begin{cases} -\dfrac{2^k}{h^{k+1}} \text{⨍}_{t_{j-1}}^{t_j} \dfrac{\mathcal{M}_{ik}^j(t,s)}{\sin^2\frac{t-s}{2}}dt, & j=m, \\ -\dfrac{2^k}{h^{k+1}} \displaystyle\int_{t_{j-1}}^{t_j} \dfrac{\mathcal{M}_{ik}^j(t,s)}{\sin^2\frac{t-s}{2}}dt, & j\neq m. \end{cases} \qquad (3.13)$$

Proof. By the definition of Cauchy principal value integral and Lemma 3.1, we have

$$\begin{aligned}
\text{⨍}_{t_{j-1}}^{t_j} \frac{\mathcal{M}_{ik}^j(t,s)}{\sin^2\frac{t-s}{2}}dt &= \text{⨍}_{t_{j-1}}^{t_j} \frac{F_i^j(t)K_s(t)}{t-s}dt \\
&= \lim_{\varepsilon\to 0}\left\{\int_{t_{j-1}}^{s-\varepsilon} \frac{F_i^j(t)K_s(t)}{t-s}dt + \int_{s+\varepsilon}^{t_j} \frac{F_i^j(t)K_s(t)}{t-s}dt\right\} \\
&= \frac{h^{k+1}}{2^{k+1}}\lim_{\varepsilon\to 0}\left\{\left(\int_{-1}^{c_m-\varepsilon} + \int_{c_m+\varepsilon}^{1}\right)\frac{F_i(\tau)K_s(\tau)}{\tau-c_m}d\tau\right\} \\
&= \frac{h^{k+1}}{2^{k+1}} \text{⨍}_{-1}^{1} \frac{F_i(\tau)K_s(\tau)}{\tau-c_m}d\tau \\
&= -\frac{h^{k+1}}{2^k}\psi_{i,i+1}(c_m). \qquad (3.14)
\end{aligned}$$

The second part of this lemma can be obtained similarly as the proof of Lemma 3.2. Here we omit it. □

Lemma 3.4. *Under the assumption of Lemma* 3.2 *and* $k>i+1$, *there holds that*

$$\psi_{ik}(c_j) = -\frac{2^k}{h^{k+1}} \text{⨎}_{t_{j-1}}^{t_j} \frac{\mathcal{M}_{ik}^j(t,s)}{(k-i)!\sin^2\frac{t-s}{2}}dt. \qquad (3.15)$$

Proof. The proof of this lemma can be obtained similarly by the change of variable $t=\hat{t}_m(\tau)$ employed as Lemma 3.2 or 3.3. □

Lemma 3.5 (See [13, Lemma 3.5]). *Suppose* $f(t)\in C^l[a,b], l\geqslant 2$. *If* $s\neq t_j$, *for any* $j=1,2,\ldots,n$, *then there holds*

$$\begin{aligned}
f(t)-f_L(t) &= \sum_{i=1}^{l-1}\sum_{k=i}^{l-1} \frac{(-1)^{i+1}f^{(k+1)}(s)}{h(i+1)!}\frac{M_{ik}^j(t,s)}{(k-i)!} \\
&\quad + \sum_{i=1}^{l-2}\frac{(-1)^{i+1}}{h(i+1)!}\frac{f^{(l)}(\xi_{ij})-f^{(l)}(s)}{(l-i-1)!}M_{i,l-1}^j(t,s)
\end{aligned}$$

$$+\frac{(-1)^l}{hl!}\tilde{M}_l^j(t), \quad \xi_{ij}\in(t_{j-1},t_j), \tag{3.16}$$

where

$$\begin{aligned}\tilde{M}_l^j(t)&=(t-t_{j-1})(t-t_j)[f^{(l)}(\eta_j)(t-t_{j-1})^{l-1}-f^{(l)}(\zeta_j)(t-t_j)^{l-1}]\\&\quad-f^{(l)}(s)M_{l-1,l-1}^j(t,s), \quad \eta_j,\zeta_j\in(t_{j-1},t_j).\end{aligned} \tag{3.17}$$

Define

$$\mathcal{H}_m(t)=f(t)-f_L(t)-\sum_{i=1}^{l-1}\sum_{k=i}^{l-1}\frac{(-1)^{i+1}f^{(k+1)}(s)}{h(i+1)!}\frac{M_{ik}^m(t,s)}{(k-i)!}, \quad t\in(t_{j-1},t_j). \tag{3.18}$$

Lemma 3.6. *Under the assumption of Theorem* 2.1, *there holds that*

$$\left|\,\,⨎_{t_{m-1}}^{t_m}\frac{\mathcal{H}_m(t)}{\sin^2\frac{t-s}{2}}dt\right|\leqslant C\gamma^{-1}(\tau)h^{l-1}. \tag{3.19}$$

Proof. Since $f(t)\in C^l[a,b]$, by Taylor expansion, we have

$$|\mathcal{H}_m^{(i)}(t)|\leqslant Ch^{l-i}, \quad i=0,1,2.$$

As

$$⨎_{t_{m-1}}^{t_m}\frac{\mathcal{H}_m(t)}{\sin^2\frac{t-s}{2}}dt=⨍_{t_{m-1}}^{t_m}\frac{\mathcal{H}_m(t)[K_s(t)-4]}{(t-s)^2}dt+4⨎_{t_{m-1}}^{t_m}\frac{\mathcal{H}_m(t)}{(t-s)^2}dt, \tag{3.20}$$

now, we estimate (3.20) term by term. For the first term, we have

$$\begin{aligned}\left|⨍_{t_{m-1}}^{t_m}\frac{\mathcal{H}_m(t)[K_s(t)-4]}{(t-s)^2}dt\right|&\leqslant\max|\mathcal{H}_m(t)|\int_{t_{m-1}}^{t_m}\frac{K_s(t)-4}{(t-s)^2}dt\\&=\max|\mathcal{H}_m(t)|\left\{⨎_{t_{m-1}}^{t_m}\frac{1}{\sin^2\frac{t-s}{2}}dt-⨎_{t_{m-1}}^{t_m}\frac{4}{(t-s)^2}dt\right\}\\&=\max|\mathcal{H}_m(t)|\left\{-2\cot\frac{s-t_{m-1}}{2}-2\cot\frac{t_m-s}{2}+\frac{4h}{(t_m-s)(s-t_{m-1})}\right\}\\&\leqslant C\gamma^{-1}(\tau)h^{l-1}.\end{aligned} \tag{3.21}$$

For the second term of (3.20), by using the identity

$$\begin{aligned}⨎_{t_{m-1}}^{t_m}\frac{\mathcal{H}_m(t)}{(t-s)^2}dt&=\frac{h\mathcal{H}_m(s)}{(s-t_{m-1})(t_m-s)}+\mathcal{H}_m'(s)\ln\frac{t_m-s}{s-t_{m-1}}\\&\quad+\int_{t_{m-1}}^{t_m}\frac{\mathcal{H}_m(t)-\mathcal{H}_m(s)-\mathcal{H}_m'(s)(t-s)}{(t-s)^2}dt.\end{aligned} \tag{3.22}$$

For the second part of (3.22), since $\mathcal{H}_m(t_m)=0$, we have

$$\left|\frac{h\mathcal{H}_m(s)}{(s-t_{m-1})(t_m-s)}\right|=\left|\frac{h[\mathcal{H}_m(s)-\mathcal{H}_m(t_m)]}{(s-t_{m-1})(t_m-s)}\right|=\left|\frac{h\mathcal{H}_m'(\xi_m)}{(s-t_{m-1})}\right|\leqslant Ch^{l-1}, \quad \xi_m\in(s,t_m), \tag{3.23}$$

$$\left|\mathcal{H}_m'(s)\ln\frac{t_m-s}{s-t_{m-1}}\right|\leqslant C|\ln\gamma(\tau)|h^{l-1}, \tag{3.24}$$

and

$$\left|\int_{t_{m-1}}^{t_m}\frac{\mathcal{H}_m(t)-\mathcal{H}_m(s)-\mathcal{H}_m'(s)(t-s)}{(t-s)^2}dt\right|=\left|\int_{t_{m-1}}^{t_m}\frac{1}{2}\mathcal{H}_m''(\eta_m)dt\right|\leqslant Ch^{l-1}, \tag{3.25}$$

where $\eta_m\in(t_{m-1},t_m)$. Combining (3.21) and (3.23)–(3.25) leads to (3.19) and the proof is finished. □

Lemma 3.7. *Under the assumption of Theorem 2.1, we have*

$$\Big|\sum_{j=1,j\neq m}^{n}\frac{(-1)^l}{hl!}\int_{t_{j-1}}^{t_j}\frac{\tilde{\mathcal{M}}_l^j(t)}{\sin^2\frac{t-s}{2}}\,dt\Big|\leqslant C\max_{t\in(t_{j-1},t_j)}\{K_s(t)\}\gamma^{-1}(\tau)\frac{h^{l-1}}{l!} \tag{3.26}$$

and

$$\Big|\sum_{i=1}^{l-2}\frac{(-1)^{i+1}}{h(i+1)!}\sum_{j=1,j\neq m}^{n}\int_{t_{j-1}}^{t_j}\frac{f^{(l)}(\xi_{ij})-f^{(l)}(s)}{(l-i-1)!}\frac{\mathcal{M}_{i,l-1}^j(t,s)}{\sin^2\frac{t-s}{2}}\,dt\Big|$$
$$\leqslant C\max_{t\in(t_{j-1},t_j)}\{K_s(t)\}\frac{h^{l-1}}{(l-1)!}(|\ln\gamma(\tau)|+|\ln h|). \tag{3.27}$$

Proof. By (3.17), we see that $|\tilde{\mathcal{M}}_l^j(t)|\leqslant Ch^{l+1}$, and thus

$$\begin{aligned}\Big|\sum_{j=1,j\neq m}^{n}\frac{(-1)^l}{hl!}\int_{t_{j-1}}^{t_j}\frac{\tilde{\mathcal{M}}_l^j(t)}{\sin^2\frac{t-s}{2}}dt\Big| &= \Big|\sum_{j=1,j\neq m}^{n}\frac{(-1)^l}{hl!}\int_{t_{j-1}}^{t_j}\frac{\tilde{\mathcal{M}}_l^j(t)K_s(t)}{(t-s)^2}dt\Big|\\ &\leqslant C\max_{t\in(t_{j-1},t_j)}\{K_s(t)\}\frac{h^l}{l!}\sum_{j=1,j\neq m}^{n}\int_{t_{j-1}}^{t_j}\frac{1}{(t-s)^2}dt\\ &\leqslant C\max_{t\in(t_{j-1},t_j)}\{K_s(t)\}\gamma^{-1}(\tau)\frac{h^{l-1}}{l!}.\end{aligned}$$

Now, we estimate (3.27) and get

$$\begin{aligned}&\Big|\sum_{j=1,j\neq m}^{n}\frac{(-1)^{l-1}}{h(l-1)!}\int_{t_{j-1}}^{t_j}(f^{(l)}(\xi_{il})-f^{(l)}(s))\frac{\mathcal{M}_{l-2,l-1}^j(t,s)}{\sin^2\frac{t-s}{2}}dt\Big|\\ &\quad=\Big|\sum_{j=1,j\neq m}^{n}\frac{(-1)^{l-1}}{h(l-1)!}\int_{t_{j-1}}^{t_j}(f^{(l)}(\xi_{il})-f^{(l)}(s))\frac{\mathcal{M}_{l-2,l-1}^j(t,s)K_s(t)}{(t-s)^2}dt\Big|\\ &\quad\leqslant C\max_{t\in(t_{j-1},t_j)}\{K_s(t)\}\frac{h^{l-1}}{(l-1)!}\sum_{j=1,j\neq m}^{n}\int_{t_{j-1}}^{t_j}\Big|\frac{1}{t-s}\Big|dt\\ &\quad\leqslant C\max_{t\in(t_{j-1},t_j)}\{K_s(t)\}(|\ln h|+|\ln\gamma(\tau)|)\frac{h^{l-1}}{(l-1)!}.\end{aligned}$$

The proof is completed. □

Lemma 3.8. *Under the assumption of Theorem 2.1 with $k=i,i+1$, there holds that*

$$T_{ik}(\tau)=4\sum_{j=-\infty}^{\infty}\phi_{ik}(2j+\tau), \tag{3.28}$$

where

$$\phi_{ik}(t)=\begin{cases}\displaystyle ⨎_{-1}^{1}\frac{\mathcal{M}_{ik}(\tau,t)}{(\tau-t)^2}d\tau, & |t|<1,\\ \displaystyle\int_{-1}^{1}\frac{\mathcal{M}_{ik}(\tau,t)}{(\tau-t)^2}d\tau, & |t|>1,\end{cases} \tag{3.29}$$

is related with the second kind of Legendre function.

Proof. By (3.10), we have

$$⨎_{-1}^{1}\frac{\mathcal{M}_{ik}(\tau,t)}{\sin^2\frac{\tau-t}{2}}d\tau=4⨎_{-1}^{1}\frac{\mathcal{M}_{ik}(\tau,t)}{(\tau-t)^2}d\tau+4\sum_{l=1}^{\infty}\int_{-1}^{1}\frac{\mathcal{M}_{ik}(\tau,t)}{(\tau-t-2l\pi)^2}d\tau$$

$$+ 4\sum_{l=1}^{\infty}\int_{-1}^{1}\frac{\mathcal{M}_{ik}(\tau,t)}{(\tau-t+2l\pi)^2}d\tau, \tag{3.30}$$

which means

$$\begin{aligned} T_{ik}(\tau) &= \sum_{i=1}^{n}\psi_{ik}(t) \\ &= 4\sum_{i=1}^{n}\phi_{ik}(2(m-i)+\tau) + 4\sum_{i=1}^{n}\sum_{l=1}^{\infty}\phi_{ik}(2(m-i-nl)+\tau) \\ &\quad + 4\sum_{i=1}^{n}\sum_{l=1}^{\infty}\phi_{ik}(2(m-i+nl)+\tau) \\ &= 4\sum_{l=-\infty}^{\infty}\sum_{i=1}^{n}\phi_{ik}(2(m-i+nl)+\tau) \\ &= 4\bigg[\phi_{ik}(\tau) + \sum_{l=1}^{\infty}\phi_{ik}(2l+\tau) + \sum_{l=1}^{\infty}\phi_{ik}(-2l+\tau)\bigg]. \end{aligned} \tag{3.31}$$

Since

$$Q_0(t) = \frac{1}{2}\log\left|\frac{1+t}{1-t}\right|,$$

then we have

$$\sum_{j=0}^{\infty}Q_0(2j+\tau) + \sum_{j=1}^{\infty}Q_0(-2j+\tau) = \frac{1}{2}\lim_{k\to\infty}\sum_{k=-n}^{k=n}\log\frac{2k+1+\tau}{2k+1-\tau} = 0.$$

By Lemmas 3.1 and 3.2 in [13], we can easily show that $T_{ik}(\tau)$ converges to certain function. □

3.1 Proof of Theorem 2.1

Proof. By Lemma 3.5, we have

$$\begin{aligned} \left(\int_{c}^{t_{m-1}} + \int_{t_m}^{c+2\pi}\right)\frac{f(t)-f_L(t)}{\sin^2\frac{t-s}{2}}dt &= \sum_{j=1,j\neq m}^{n}\int_{t_{j-1}}^{t_j}\frac{f(t)-f_L(t)}{(t-s)^2\sin^2\frac{t-s}{2}}dt \\ &= \sum_{i=1}^{l-1}\sum_{k=i}^{l-1}\frac{(-1)^{i+1}f^{(k+1)}(s)}{h(i+1)!(k-i)!}\sum_{j=1,j\neq m}^{n}\int_{t_{j-1}}^{t_j}\frac{\mathcal{M}_{ik}^{j}(t,s)}{\sin^2\frac{t-s}{2}}dt \\ &\quad + \sum_{i=1}^{l-2}\frac{(-1)^{i+1}}{h(i+1)!}\sum_{j=1,j\neq m}^{n}\int_{t_{j-1}}^{t_j}\frac{f^{(l)}(\xi_{il})-f^{(l)}(s)}{(l-i-1)!}\frac{\mathcal{M}_{i,l-1}^{j}(t,s)}{\sin^2\frac{t-s}{2}}dt \\ &\quad + \sum_{j=1,j\neq m}^{n}\frac{(-1)^n}{hl!}\int_{t_{j-1}}^{t_j}\frac{\tilde{\mathcal{M}}_{l}^{j}(t)}{\sin^2\frac{t-s}{2}}\,dt. \end{aligned}$$

By the definition of $\mathcal{H}_j(t)$ in (3.18), we have

$$\fint_{t_{m-1}}^{t_m}\frac{f(t)-f_L(t)}{\sin^2\frac{t-s}{2}}\,dt = \fint_{t_{m-1}}^{t_m}\frac{\mathcal{H}_m(t)}{\sin^2\frac{t-s}{2}}\,dt + \sum_{i=1}^{l-1}\sum_{k=i}^{l-1}\frac{(-1)^{i+1}f^{(k+1)}(s)}{h(i+1)!(k-i)!}\fint_{t_{m-1}}^{t_m}\frac{\mathcal{M}_{ik}^{m}(t,s)}{\sin^2\frac{t-s}{2}}\,dt. \tag{3.32}$$

Then we have

$$\begin{aligned} \int_{c}^{c+2\pi} = \frac{f(t)-f_L(t)}{\sin^2\frac{t-s}{2}}dt &= \sum_{i=1}^{l-1}\sum_{k=i}^{l-1}\frac{(-1)^{i+1}h^k f^{(k+1)}(s)}{(i+1)!(k-i)!}\sum_{j=1}^{n}\psi_{ik}(\tau) + \mathcal{R}_n(s) \\ &= \sum_{i=1}^{l-1}\frac{h^i}{2^i}f^{(i+1)}(s)a_i(\tau) + R_n(s), \end{aligned} \tag{3.33}$$

where

$$a_i(\tau)=\begin{cases}\displaystyle\sum_{k=i-1}^{i}\frac{(-1)^{k+1}}{(k+1)!}T_{ki}(\tau), & i>1,\\ \displaystyle\frac{1}{2}T_{ki}(\tau), & i=1,\end{cases} \tag{3.34}$$

$$\mathcal{R}_n(s)=\mathcal{R}_n^{(1)}(s)+\mathcal{R}_n^{(2)}(s) \tag{3.35}$$

and

$$\begin{aligned}\mathcal{R}_n^{(1)}(s)&=\fint_{t_{m-1}}^{t_m}\frac{\mathcal{H}_m(t)}{\sin^2\frac{t-s}{2}}dt,\\ \mathcal{R}_n^{(2)}(s)&=\frac{(-1)^{l-1}}{h(l-1)!}\sum_{j=1,j\neq m}^{n}\int_{t_{j-1}}^{t_j}\frac{f^{(l)}(\xi_{ij})-f^{(l)}(s)}{(l-i-1)!}\frac{\mathcal{M}_{i,l-1}^j(t,s)}{\sin^2\frac{t-s}{2}}dt\\ &\quad+\sum_{j=1,j\neq m}^{n}\frac{(-1)^l}{hl!}\int_{t_{j-1}}^{t_j}\frac{\tilde{\mathcal{M}}_l^j(t)}{\sin^2\frac{t-s}{2}}dt.\end{aligned} \tag{3.36}$$

By Lemmas 3.6 and 3.7, we have

$$|\mathcal{R}_n(s)|\leqslant C\max_{t\in(c,c+2\pi)}\{K_s(t)\}(|\ln h|+\gamma^{-1}(\tau))h^{l-1}.$$

The proof is completed. □

By a straight calculation, the explicit expression of $T_{ik}(\tau)$ can be written by the series of Clausen functions with very preliminary proof and involves tedious details. Here we sketch it. By the definition of (1.7), we can easily get

$$a_1(\tau)=-\frac{1}{2}T_{11}(\tau)=\ln 2\cos\frac{\pi\tau}{2};$$

when $a_1(\tau)=0$, then we get the superconvergence point $\tau=\pm\frac{2}{3}$.

4 Extrapolation method

In the above sections, we have proved that the error functional of the trapezoidal rule has the following asymptotic expansion

$$E_n(f)=\sum_{i=1}^{l-1}\frac{h^i}{2^i}f^{(i+1)}(s)a_i(\tau)+R_n(s). \tag{4.1}$$

It is easy to see that the error functional depends on the value of $a_i(\tau)$.

Now we present algorithm for the given s. Assume that there exists positive integer n_0 such that

$$m_0:=\frac{n_0(s-a)}{b-a}$$

is a positive number. We first partition $[a,b]$ into n_0 equal subinterval to get a mesh denoted by Π_1 with mesh size $h_1=(b-a)/n_0$. Then we refine Π_1 to get mesh Π_2 with mesh size $h_2=h_1/2$. In this way, we get a series of meshes $\{\Pi_j\}(j=1,2,\ldots)$ in which Π_j is refined from Π_{j-1} with mesh size denoted by h_j. The extrapolation scheme is presented in Table 1.

For a given $\tau\in(-1,1)$, define

$$s_j=s+\frac{\tau+1}{2}h_j,\quad j=1,2,\ldots \tag{4.2}$$

and

$$T(h_j)=I_{2^{j-1}n_0}(f,s_j). \tag{4.3}$$

Table 1 Extrapolation scheme of $T_i^{(j)}$

$T(h_1)=T_1^{(1)}$				
$T(h_2)=T_1^{(2)}$	$T_2^{(1)}$			
$T(h_3)=T_1^{(3)}$	$T_2^{(2)}$	$T_3^{(1)}$		
$T(h_4)=T_1^{(4)}$	$T_2^{(3)}$	$T_3^{(2)}$	$T_4^{(1)}$	
$T(h_5)=T_1^{(5)}$	$T_2^{(4)}$	$T_3^{(3)}$	$T_4^{(2)}$	$T_5^{(1)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

We present the following extrapolation algorithm:

Step One: Compute $T_1^{(j)}=T(h_j), j=1,\dots,m$.

Step Two: Compute $T_i^{(j)}=T_{i-1}^{(j+1)}+\frac{T_{i-1}^{(j+1)}-T_{i-1}^{(j)}}{2^{i-1}-1}, i=2,\dots,m, j=1,\dots,m-i$.

Theorem 4.1. *Under the asymptotic expansion of Theorem* 2.1*, for a given* τ *and the series of meshes* s_j *defined by* (4.2)*, we have*

$$|I(f,s)-T_i^{(j)}|\leqslant Ch^i \tag{4.4}$$

and a posteriori asymptotic error estimate is given by

$$\left|\frac{T_{i-1}^{(j+1)}-T_{i-1}^{(j)}}{2^{i-1}-1}\right|\leqslant Ch^{i-1}.$$

Proof. It is similar to the proof of Theorem 4.1 in [13]. □

5 Numerical example

In this section, computational results are reported to confirm our theoretical analysis.

Example 5.1. We consider the finite-part integral with $f(t)=\cos t, a=-\pi, b=\pi$. Obviously, the integrand function $f(t)$ is smooth enough and by (1.2), we have

$$⨎_{-\pi}^{\pi}\frac{f(t)}{\sin^2\frac{t-s}{2}}dt=-4\pi\cos s.$$

In Tables 2 and 3, we have presented the superconvergence phenomenon with $s=t_{[n/2]}+(1+\tau)h/2$ and $s=a+(1+\tau)h/2$. When the local coodinate takes $\pm\frac{2}{3}$ the convergence rate is $O(h^2)$; while for the non-superconvergence point $\tau\neq\pm\frac{2}{3}$, the convergence rate is $O(h)$, which agrees with our theoretical analysis very well.

Example 5.2. We consider the finite-part integral with $f(t)=1+2\cos t+2\cos 2t, a=-\pi, b=\pi$. Obviously, the integrand function $f(t)$ is smooth enough and by (1.2), we get the analysis solution

$$⨎_{-\pi}^{\pi}\frac{f(t)}{\sin^2\frac{t-s}{2}}dt=-8\pi(\cos s+2\cos 2s).$$

We choose the series $s_j=s+\frac{\tau+1}{2}h_j$ to approximate the singular point s with $s=-\pi/2$ and $s=-\pi$, respectively. From Tables 4 and 7, we know that with $\tau=0,\pm2/3$ or $\tau=1/2$, there are the same convergence rate with $O(h)$, so it is the same to choose any τ. In Tables 5 and 6, we choose $\tau=0$, the error estimate and a posterior error estimate with convergence order h^2, h^3 and h^4 are shown which agree with the extrapolation algorithm very well.

For the case of $s=a$ we still choose $\tau=0$ to get the error estimate and a posterior error estimate with convergence order h^2, h^3 and h^4 in Tables 8 and 9.

For a given s, we can find the starting meshes n_0 theoretically. But in actual computation, if n_0 is too big, the extrapolation algorithm is not proper to be adopted. For example, with $s=\pi/\sqrt{2}$, we

Table 2 Errors of the trapezoidal rule with $s = t_{[n/2]} + (1+\tau)h/2$

n	$\tau = 0$	$\tau = -2/3$	$\tau = 2/3$	$\tau = 1/2$
32	$-5.8085e-1$	$-3.9180e-2$	$-4.3528e-2$	$-3.1165e-1$
64	$-2.8182e-1$	$-9.9484e-3$	$-1.0527e-2$	$-1.4611e-1$
128	$-1.3856e-1$	$-2.5055e-3$	$-2.5798e-3$	$-7.0564e-2$
512	$-6.8673e-2$	$-6.2862e-4$	$-6.3804e-4$	$-3.4655e-2$
1024	$-3.4182e-2$	$-1.5742e-4$	$-1.5861e-4$	$-1.7170e-2$
2048	$-1.7052e-2$	$-3.9345e-5$	$-3.9493e-5$	$-8.5456e-3$

Table 3 Errors of the trapezoidal rule with $s = a + (1+\tau)h/2$

n	$\tau = 0$	$\tau = -2/3$	$\tau = 2/3$	$\tau = 1/2$
32	5.8085e−1	3.9180e−2	4.3528e−2	3.1165e−1
64	2.8182e−1	9.9484e−3	1.0527e−2	1.4611e−1
128	1.3856e−1	2.5055e−3	2.5798e−3	7.0564e−2
512	6.8673e−2	6.2862e−4	6.3804e−4	3.4655e−2
1024	3.4182e−2	1.5742e−4	1.5861e−4	1.7170e−2
2048	1.7052e−2	3.9345e−5	3.9493e−5	8.5456e−3

Table 4 Errors of the trapezoidal rule $s_j = s + (\tau+1)h_j/2$

n	0	2/3	−2/3	1/2
32	8.1836e+0	7.5238e+0	1.4834e+0	8.5833e+0
64	3.7710e+0	2.9101e+0	5.8207e−1	3.6292e+0
128	1.7978e+0	1.2409e+0	2.4898e−1	1.6433e+0
256	8.7613e−1	5.6713e−1	1.1373e−1	7.7836e−1
512	4.3227e−1	2.7027e−1	5.4144e−2	3.7831e−1
1024	2.1467e−1	1.3182e−1	2.6388e−2	1.8643e−1

Table 5 Extrapolation errors of the trapezoidal rule $s_j = s + (\tau+1)h_j/2$

n	0	h^2-extra	h^3-extra	h^4-extra
32	8.1836e+0			
64	3.7710e+0	−6.4164e−1		
128	1.7978e+0	−1.7536e−1	−1.9940e−2	
256	8.7613e−1	−4.5544e−2	−2.2699e−3	2.5439e−4
512	4.3227e−1	−1.1587e−2	−2.6881e−4	1.7053e−5
1024	2.1467e−1	−2.9214e−3	−3.2661e−5	1.0752e−6

Table 6 A posterior error of the trapezoidal rule $s_j = s + (\tau+1)h_j/2$

n	0	h^2-extra	h^3-extra	h^4-extra
32				
64	4.4126e+0			
128	1.9732e+0	−1.5542e−1		
256	9.2167e−1	−4.3274e−2	−2.5243e−3	
512	4.4386e−1	−1.1319e−2	−2.8587e−4	1.5822e−5
1024	2.1760e−1	−2.8887e−3	−3.3736e−5	1.0652e−6

cannot find the proper starting meshes. There are many methods to solve the problem. One method is by moving the starting meshes a little, then making the singular point be located at the mesh point. In fact, it is not difficult to extend our methods to the quasi-uniform meshes.

Table 7 Errors of the trapezoidal rule $s_j = s + (\tau + 1)h_j/2$

n	0	2/3	−2/3	1/2
32	−4.5518e+0	−2.9711e+0	−6.2152e−1	−4.0474e+0
64	−1.9715e+0	−7.4038e−1	−1.6037e−1	−1.4303e+0
128	−9.0311e−1	−1.8405e−1	−4.0670e−2	−5.6247e−1
256	−4.3016e−1	−4.5836e−2	−1.0237e−2	−2.4275e−1
512	−2.0964e−1	−1.1434e−2	−2.5676e−3	−1.1173e−1
1024	−1.0345e−1	−2.8552e−3	−6.4289e−4	−5.3452e−2

Table 8 Extrapolation errors of the trapezoidal rule $s_j = s + (\tau + 1)h_j/2$

n	0	h^2-extra	h^3-extra	h^4-extra
32	−4.5518e+0			
64	−1.9715e+0	6.0874e−1		
128	−9.0311e−1	1.6531e−1	1.7498e−2	
256	−4.3016e−1	4.2802e−2	1.9665e−3	−2.5230e−4
512	−2.0964e−1	1.0874e−2	2.3104e−4	−1.6885e−5
1024	−1.0345e−1	2.7394e−3	2.7950e−5	−1.0634e−6

Table 9 A posterior error of trapezoidal rule $s_j = s + (\tau + 1)h_j/2$

n	0	h^2-extra	h^3-extra	h^4-extra
32				
64	−2.5803e+0			
128	−1.0684e+0	1.4781e−1		
256	−4.7296e−1	4.0835e−2	2.2188e−3	
512	−2.2051e−1	1.0643e−2	2.4793e−4	−1.5694e−5
1024	−1.0619e−1	2.7115e−3	2.9013e−5	−1.0548e−6

Table 10 Errors of the trapezoidal rule $s_j = s + (\tau + 1)h_j/2$

n	0	2/3	−2/3	1/2
32	−4.3141e+0	−1.1288e+1	−1.8853e+0	−8.8513e+0
64	−2.5250e+0	−6.0289e+0	−1.1030e+0	−4.8591e+0
128	−1.3496e+0	−3.0879e+0	−5.9072e-1	−2.5222e+0
256	−6.9584e−1	−1.5594e+0	−3.0503e−1	−1.2822e+0
512	−3.5308e−1	−7.8321e−1	−1.5491e−1	−6.4613e−1
1024	−1.7782e−1	−3.9243e−1	−7.8051e−2	−3.2429e−1

Example 5.3. We still consider $f(t) = 1+2\cos t+2\cos 2t, a = -\pi, b = \pi$, for the case of quasi-uniform meshes, we consider $s = -\pi/\sqrt{2}$, in order to make sure that s is located at the meshes point, we can do it in such a way that with two meshes shorter or longer at the end of the interval by moving the starting meshes a little. Then we refine the quasi-uniform mesh and get the results as follows.

Table 10 is similar to Table 4, which has the same convergence rate with $O(h)$. From Tables 11 and 12, we choose $\tau = 0$ and the error estimate and a posterior error estimate with convergence order being h^2, h^3 and h^4 respectively, which agree with the algorithm very well.

6 Concluding remarks

In this paper, we have shown, both theoretically and numerically,that the main part of error function of the trapezoidal rule has the asymptotic expansion of (1.6). Numerical experiment has shown that the

Table 11 Extrapolation errors of the trapezoidal rule $s_j = s + (\tau + 1)h_j/2$

n	0	h^2-extra	h^3-extra	h^4-extra
32	−4.3141e+0			
64	−2.5250e+0	−7.3601e−1		
128	−1.3496e+0	−1.7410e−1	1.3198e−2	
256	−6.9584e−1	−4.2095e−2	1.9073e−3	2.9432e−4
512	−3.5308e−1	−1.0334e−2	2.5371e−4	1.7484e−5
1024	−1.7782e−1	−2.5589e−3	3.2626e−5	1.0435e−6

Table 12 A posterior error of the trapezoidal rule $s_j = s + (\tau + 1)h_j/2$

n	0	h^2-extra	h^3-extra	h^4-extra
32				
64	−1.7890e+0			
128	−1.1755e+0	−1.8730e−1		
256	−6.5374e−1	−4.4003e−2	1.6129e−3	
512	−3.4275e−1	−1.0587e−2	2.3622e−4	1.8455e−5
1024	−1.7526e−1	−2.5915e−3	3.1583e−5	1.0961e−6

special function $a_i(\tau)$ has a big influence on the convergence rate. Moreover, by the extrapolation method, we not only obtain a high order of accuracy, but also derive a posteriori error estimate conveniently.

The singular integral equations [19] and hypersingular integral equations [20,21], are presently encountered in a wide range of nonlinear mathematical models. How to solve them efficiently and exactly is an important problem in numerical mathematics, especially in boundary element methods. There are many applications to solve physical problems. A variety of special numerical quadrature methods for solving the integral in such equations have also been formulated. The results in this paper show a possible way to improve the accuracy for Hadamard finite-part integral by using the extrapolation methods. It is also convenient to get a posteriori error estimate, which means that we are sure to have a satisfactory approximation by an adaptive process.

Acknowledgements The work of Li was supported by National Natural Science Foundation of China (Grant Nos. 11101247 and 11201209), Shandong Provincial Natural Science Foundation of China (Grant No. ZR2011AQ020) and a project of Shandong Province Higher Educational Science and Technology Program (Grant No. J11LE08). The work of Zhang was supported by National Natural Science Foundation of China (Grant No. 11101317). The work of Yu was supported by National Basic Research Program of China (Grant No. 2005CB321701) and the Reward Fund of CAS for National Prize.

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