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A collocation scheme for a certain Cauchy singular integral equation based on the superconvergence analysis

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ABSTRACT

In this paper, we investigate the composite midpoint rule for the evaluation of Cauchy principal value integral in an interval and place the key point on its *pointwise superconvergence phenomenon*. The error expansion of the rule is obtained, which shows that the superconvergence phenomenon occurs at the points of each subinterval whose local coordinate is the zeros of some function. Then, by applying the midpoint rule to approximate the Cauchy principal value integral and choosing the superconvergence points as the collocation points, we obtain a collocation scheme for solving a certain Cauchy singular integral equation. The more interesting thing is that the coefficient matrix of the resulting linear system possesses some good properties, from which we obtain an optimal error estimate. Finally, some numerical examples are provided to validate the theoretical analysis.

1. Introduction

Consider the Cauchy singular integral equation

$$\fint_a^b \frac{u(x)}{x-s}\,dx = f(s), \tag{1}$$

where $f(s)$ is a given function and $u(x)$ the density function to be determined. Throughout the paper, $\fint$ denotes the Cauchy principal value integral or Hilbert transform. This integral has many different but mathematically equivalent definitions, among which we adopt the following definition

$$\fint_a^b \frac{u(x)}{x-s}\,dx := \lim_{\epsilon\to 0}\left\{\int_a^{s-\epsilon} \frac{u(x)}{x-s}\,dx + \int_{s+\epsilon}^b \frac{u(x)}{x-s}\,dx\right\} \quad s\in(a,b). \tag{2}$$

The density function $u(x)$ is said to be Cauchy principal value integrable with respect to the weight $(x-s)^{-1}$ if the limit on the right-hand side of (2) exists. A sufficient condition for $u(x)$ to be Cauchy principal value integrable is that $u(x)$ is Hölder continuous.

Integrals of this kind are widely used in many areas of mathematical physics in terms of boundary integral equations, such as potential theory, elasticity problems as well as electromagnetic scattering [6,29]. Numerous work has been devoted to developing efficient quadrature formulas, such as the Gaussian method [10,14,19], the (composite) Newton–Cotes method

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[11,17,21,22] and the modified version [2,9], spline approximation [8,12,25] and some other methods [7,13]. The Newton–Cotes method is a commonly used one due to its ease of implementation and flexibility of mesh.

It is well known that if the integrand function is approximated by a piecewise polynomial interpolant of degree k, the accuracy of the Newton–Cotes method for Riemann integrals is $O(h^{k+1})$ for odd k and $O(h^{k+2})$ for even k. However, the rule is less accurate for singular integrals due to the singularity of the integrand. For example, the corresponding result for Cauchy principal value integrals is generally only $O(h^{k+1})$ [11,17]. The accuracy may be higher if the singular point s coincides with some *a priori* known point. We referred to this as *the pointwise superconvergence phenomenon* of Newton–Cotes method for singular integrals. The superconvergence phenomenon for Hadamard finite-part integrals was firstly studied in [26,27], and then a general result was given in [28] for $k \geqslant 1$. Moreover, the general (composite) Newton–Cotes rules were studied for Hadamard finite-part integrals when the singular point coincides with an element junction point [18]. Recently, the superconvergence of Newton–Cotes rules was generalized for Cauchy principal value integrals in [22], and the accuracy can reach $O(h^{k+2})$ for even k when the singular point s coincides with the so-called "superconvergence points".

The connections of the model problem (1) with classical boundary value problems of potential theory, with boundary integral equations, and with the situation of a region bounded by a curve Γ, we refer to [6,20,24] and the references therein. A great deal of effort has been expended in the development of numerical techniques for the approximation solution of Cauchy singular integral equations. Ioakimidis [15] suggested the successive approximation method to solve the airfoil equation, then this method was applied for Cauchy singular integral equation of the first kind in the general case [23]. Abdou and Nasr [1] solved the Fredholm integral equation with Cauchy kernel by removing the singularity and expanding the solution in terms of orthogonal polynomials. Bonis and Laurita [4] proposed a quadrature type method based on a Gaussian rule for a class of systems of Cauchy singular integral equations with constant coefficients, and proved the convergence and stability of the method in weighted L^2 spaces. Kim [16] solved the Cauchy singular integral equations by using Gaussian quadrature and choosing the zeros of the Chebyshev polynomials of the first and second kinds as the collocation points. Chakrabarti and Berghe [5] proposed an approximation method using polynomial approximation of degree n and the zeros of Chebyshev polynomials as collocation points. Chandler [6] studied the midpoint collocation method when the grid is non-uniform. However, these papers did not reflect comprehensively the order of the convergence rate achieved with the methods in practice.

This paper has two purposes. The first is to investigate the superconvergence of the composite midpoint rule, one of the lowest order Newton–Cotes rules, for Cauchy principal value integrals. The second is to construct a simple collocation scheme for solving certain Cauchy singular integral equations, based on the superconvergence result of the midpoint rule. In the process, some surprising results are explored, such as symmetry, Toeplitz type, M-type and strictly diagonally dominant. With these, we finally obtain the solvability and the striking optimal error estimate of the scheme.

The rest of this paper is organized as follows. Section 2 discusses the general (composite) midpoint rule and presents the main superconvergence results. Section 3 proposes a collocation scheme for a certain Cauchy singular integral equation, gives an optimal error estimate. Section 4 presents several numerical examples to confirm our theoretical analysis. Some remarks conclude the paper in Section 5.

2. The composite midpoint rule and its superconvergence result

For simplicity of exposition, we confine ourselves to the case where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ is a uniform mesh of $[a,b]$ with the mesh size $h = (b-a)/n$. Let $u_0^I(x)$ be the piecewise constant interpolation of $u(x)$, defined by

$$u_0^I(x) = \sum_{i=0}^{n-1} u(\hat{x}_i)\varphi_i(x),$$

where $\hat{x}_i = (x_i + x_{i+1})/2$ and

$$\varphi_i(x) = \begin{cases} 1 & \text{on } [x_i, x_{i+1}], \\ 0 & \text{otherwise}. \end{cases} \tag{3}$$

Replacing $u(x)$ in (1) with $u_0^I(x)$ gives the composite midpoint rule

$$\mathcal{Q}_h u(s) = \fint_a^b \frac{u_0^I(x)}{x-s}\,dx = \sum_{i=0}^{n-1}\omega_i(s)u(\hat{x}_i) = \fint_a^b \frac{u(x)}{x-s}\,dx - \mathcal{E}(s;u), \tag{4}$$

where $\mathcal{E}(s;u)$ denotes the error functional and

$$\omega_i(s) = \fint_a^b \frac{\varphi_i(x)}{x-s}\,dx = \ln\left|\frac{x_{i+1}-s}{x_i-s}\right|. \tag{5}$$

Recall that $Q_k(x)$ be the second function associated with Legendre polynomial $P_k(x)$,

$$Q_k(x) = \begin{cases} \frac{1}{2}\fint_{-1}^{1}\frac{P_k(t)}{x-t}\,dt & |x| < 1, \\ \frac{1}{2}\int_{-1}^{1}\frac{P_k(t)}{x-t}\,dt & |x| > 1, \end{cases} \tag{6}$$

$$Q_0(x) = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|, \quad Q_1(x) = xQ_0(x) - 1,$$
$$Q_{k+1}(x) = \frac{2k+1}{k+1}xQ_k(x) - \frac{k}{k+1}Q_{k-1}(x) \quad k = 1, 2, \ldots \tag{7}$$

The classical identity [3]

$$Q_k(x) = \frac{1}{2^{k+1}}\int_{-1}^{1}\frac{(1-t^2)^k}{(x-t)^{k+1}}\,dt \quad |x| > 1, \;\; k = 0, 1, 2, \ldots \tag{8}$$

leads to

$$|Q_k(x)| = \frac{1}{2^{k+1}}\left|\int_{-1}^{1}\frac{(1-t^2)^k}{(x-t)^{k+1}}\,dt\right| \leqslant \frac{C}{(|x|-1)^{k+1}} \quad |x| > 1. \tag{9}$$

Throughout this paper, C will denote a generic positive constant which is independent of the mesh parameter h and the singular point s. Moreover, we assume $s \in (x_m, x_{m+1})$ with some fixed m, i.e., $s = x_m + (1+\tau)h/2$, where $\tau \in (-1, 1)$ is local coordinate of the singular point s. Define the operator $\mathcal{W}$ by

$$\mathcal{W}(f;\tau) := f(\tau) + \sum_{i=1}^{\infty}[f(2i+\tau) + f(-2i+\tau)] \quad \tau \in (-1, 1). \tag{10}$$

Obviously, $\mathcal{W}$ is a linear operator on f.

The remaining part of this section is devoted to Theorem 1 which is based on the following lemmas.

Lemma 1 [30]. *Let $\mathcal{W}$ be a linear operator defined in (10), it holds*

$$\mathcal{W}(Q_1;\tau) = -\ln\left(2\cos\frac{\tau\pi}{2}\right) \quad |\tau| < 1.$$

Define

$$\mathcal{I}_{n,i}(s) = \begin{cases} \int_{x_i}^{x_{i+1}} \frac{x-\hat{x}_i}{x-s}\,dx & i \neq m, \\ ⨍_{x_m}^{x_{m+1}} \frac{x-\hat{x}_i}{x-s}\,dx & i = m. \end{cases} \tag{11}$$

Lemma 2. *It holds that*

$$\sum_{i=0}^{n-1}\mathcal{I}_{n,i}(s) = h\ln\left(2\cos\frac{\tau\pi}{2}\right) + h\left[\sum_{i=m+1}^{\infty}Q_1(2i+\tau) + \sum_{i=n-m}^{\infty}Q_1(-2i+\tau)\right],$$

where

$$\left|\sum_{i=m+1}^{\infty}Q_1(2i+\tau) + \sum_{i=n-m}^{\infty}Q_1(-2i+\tau)\right| \leqslant C\eta(s)h \tag{12}$$

and

$$\eta(s) = \max\left\{\frac{1}{b-s}, \frac{1}{s-a}\right\}. \tag{13}$$

Proof. Let $c_i = 2(s-x_i)/h - 1 = 2(m-i) + \tau$. If $i = m$, the definition of the Cauchy principal value integral and the $Q_1(x)$ lead to

$$\mathcal{I}_{n,m}(s) = \lim_{\epsilon\to 0}\left\{\int_{x_m}^{s-\epsilon}\frac{x-\hat{x}_m}{x-s}\,dx + \int_{s+\epsilon}^{x_{m+1}}\frac{x-\hat{x}_m}{x-s}\,dx\right\} = \frac{h}{2}\lim_{\epsilon\to 0}\left\{\int_{-1}^{c_m-2\epsilon/h}\frac{t}{t-c_m}\,dt + \int_{c_m+2\epsilon/h}^{1}\frac{t}{t-c_m}\,dt\right\}$$
$$= \frac{h}{2}⨍_{-1}^{1}\frac{t}{t-c_m}\,dt = -hQ_1(c_m). \tag{14}$$

Similarly,

$$\mathcal{I}_{n,i}(s) = -hQ_1(c_i), \quad i \neq m.$$

Lemma 1 shows that,

$$\sum_{i=0}^{n-1}\mathcal{I}_{n,i}(s) = -h\sum_{i=0}^{n-1}Q_1(c_i) = -h\sum_{i=0}^{n-1}Q_1(2(m-i)+\tau) = -h\mathcal{W}(Q_1;\tau) + h\left[\sum_{i=m+1}^{\infty}Q_1(2i+\tau) + \sum_{i=n-m}^{\infty}Q_1(-2i+\tau)\right]$$
$$= h\ln\left(2\cos\frac{\tau\pi}{2}\right) + h\left[\sum_{i=m+1}^{\infty}Q_1(2i+\tau) + \sum_{i=n-m}^{\infty}Q_1(-2i+\tau)\right].$$

Moreover, (9) implies that

$$\left|\sum_{i=m+1}^{\infty}Q_1(2i+\tau) + \sum_{i=n-m}^{\infty}Q_1(-2i+\tau)\right| \leqslant \sum_{i=m+1}^{\infty}\frac{C}{(|2i+\tau|-1)^2} + \sum_{i=n-m}^{\infty}\frac{C}{(|2i-\tau|-1)^2} \leqslant C\left[\frac{1}{m+1}+\frac{1}{n-m}\right] \leqslant C\eta(s)h.$$

This concludes the proof. □

Lemma 3. *It holds that*

$$\sum_{i=0,i\neq m}^{n-1}[u'(\eta_i)-u'(s)]\mathcal{I}_{n,i}(s) \leqslant \begin{cases} C\gamma^{-2}(h,s)h^{1+\alpha} & u\in C^{1+\alpha}[a,b],\\ C\gamma^{-2}(h,s)h^2|\ln h| & u\in C^2[a,b],\\ C\gamma^{-2}(h,s)h^2 & u\in C^{2+\alpha}[a,b],\end{cases} \tag{15}$$

where $\eta_i\in[x_i,x_{i+1}],\ 0<\alpha<1,$

$$\gamma(h,s) = \min_{0\leqslant i\leqslant n}\frac{|s-x_i|}{h} = \frac{1-|\tau|}{2}. \tag{16}$$

Proof. If $i\neq m$, integrating by parts on $\mathcal{I}_{n,i}(s)$ leads to

$$\mathcal{I}_{n,i}(s) = \frac{h}{2}(\ln|x_i-s| + \ln|x_{i+1}-s|) - \int_{x_i}^{x_{i+1}}\ln|x-s|\,dx,$$

which is actually the error of the trapezoidal rule for certain Riemann integral on $[x_i,x_{i+1}]$. Therefore, there exists $\tilde{x}_i\in[x_i,x_{i+1}]$ such that

$$\mathcal{I}_{n,i}(s) = \frac{h^3}{12(\tilde{x}_i-s)^2}.$$

If $u(x)\in C^{1+\alpha}[a,b]$ $(0<\alpha\leqslant 1)$, we have

$$\left|\sum_{i=0,i\neq m}^{n-1}[u'(\eta_i)-u'(s)]\mathcal{I}_{n,i}(s)\right| \leqslant \sum_{i=0,i\neq m}^{n-1}\frac{h^3|\eta_i-s|^\alpha}{12(\tilde{x}_i-s)^2} \leqslant C\left[\sum_{i=0}^{m-1}\frac{h^3(s-x_i)^\alpha}{12(s-x_{i+1})^2} + \sum_{i=m+1}^{n-1}\frac{h^3(x_{i+1}-s)^\alpha}{12(s-x_i)^2}\right]. \tag{17}$$

Since $s = x_m + (1+\tau)h/2$ $(-1<\tau<1)$, we have

$$\sum_{i=0}^{m-1}\frac{h^3(s-x_i)^\alpha}{(s-x_{i+1})^2} \leqslant \sum_{i=0}^{m-1}\frac{h^{3+\alpha}+h^3(s-x_{i+1})^\alpha}{(s-x_{i+1})^2} \leqslant h^{1+\alpha}\sum_{i=0}^{m-1}\frac{1+(m-i-1+\frac{1+\tau}{2})^\alpha}{(m-i-1+\frac{1+\tau}{2})^2} \leqslant \begin{cases}\frac{Ch^{1+\alpha}}{(1+\tau)^2} & 0<\alpha<1,\\ \frac{Ch^2|\ln h|}{(1+\tau)^2} & \alpha=1.\end{cases} \tag{18}$$

Similarly,

$$\sum_{i=m+1}^{n-1}\frac{h^3(x_{i+1}-s)^\alpha}{(s-x_i)^2} \leqslant \begin{cases}\frac{Ch^{1+\alpha}}{(1-\tau)^2} & 0<\alpha<1,\\ \frac{Ch^2|\ln h|}{(1-\tau)^2} & \alpha=1.\end{cases} \tag{19}$$

Therefore, the first two bounds in (15) can be directly obtained from (17)–(19). If $u(x)\in C^{2+\alpha}[a,b](0<\alpha<1)$, a similar approach leads to

$$\left|\sum_{i=0,i\neq m}^{n-1}[u'(\eta_i)-u'(s)]\mathcal{I}_{n,i}(s)\right| \leqslant C\left[\sum_{i=0}^{m-1}\frac{h^3(s-x_i)^{1+\alpha}}{12(s-x_{i+1})^2} + \sum_{i=m+1}^{n-1}\frac{h^3(x_{i+1}-s)^{1+\alpha}}{12(s-x_i)^2}\right], \tag{20}$$

then,

$$\sum_{i=0}^{m-1}\frac{h^3(s-x_i)^{1+\alpha}}{(s-x_{i+1})^2} \leqslant \sum_{i=0}^{m-1}\frac{h^{3+\alpha}+h^3(s-x_{i+1})^{1+\alpha}}{(s-x_{i+1})^2} \leqslant h^{1+\alpha}\sum_{i=0}^{m-1}\frac{1+(m-i-1+\frac{1+\tau}{2})^{1+\alpha}}{(m-i-1+\frac{1+\tau}{2})^2} \leqslant \frac{Ch^2}{(1+\tau)^2}. \tag{21}$$

Similarly,

$$\sum_{i=m+1}^{n-1}\frac{h^3(x_{i+1}-s)^\alpha}{(s-x_i)^2}\leqslant\frac{Ch^2}{(1-\tau)^2}. \tag{22}$$

Combining (20) and (21) with (22) yields the last bound in (15), which concludes the proof. □

Theorem 1. *Let $\mathcal{Q}_h u(s)$ be computed by (4) and (5) with a uniform mesh. Then for $s\neq x_i$ $(1\leqslant i\leqslant n)$ and $-1<\tau<1$, it holds*

$$\mathcal{E}(s;u)=hu'(s)\ln\left(2\cos\frac{\tau\pi}{2}\right)+\mathcal{R}(s), \tag{23}$$

where

$$|\mathcal{R}(s)|\leqslant\begin{cases}C[\gamma^{-2}(h,s)+\eta(s)h^{1-\alpha}]h^{1+\alpha} & u(x)\in C^{1+\alpha}[a,b],\\ C[\gamma^{-2}(h,s)+|\ln h|+\eta(s)]h^2 & u(x)\in C^2[a,b],\\ C[\gamma^{-2}(h,s)+\eta(s)]h^2 & u(x)\in C^{2+\alpha}[a,b].\end{cases} \tag{24}$$

Remark 1. It is natural to propose a modified version of the midpoint rule $\mathcal{Q}_h u(s)$,

$$\tilde{\mathcal{Q}}_h u(s)=\mathcal{Q}_h u(s)+hu'(s)\ln\left(2\cos\frac{\tau\pi}{2}\right), \tag{25}$$

whose error can be just estimated by $\mathcal{R}(s)$.

Proof of Theorem 1. Taylor expansion shows that there exists $\xi_i\in(x_i,x_{i+1})$ such that

$$u(x)-u(\hat{x}_i)=u'(\xi_i)(x-\hat{x}_i)\quad x\in[x_i,x_{i+1}].$$

The means value theorem of integration implies

$$\begin{aligned}\sum_{i=0,i\neq m}^{n-1}\int_{x_i}^{x_{i+1}}\frac{u(x)-u(\hat{x}_i)}{x-s}dx&=-\sum_{i=0}^{m-1}\int_{x_i}^{x_{i+1}}\frac{u'(\xi_i)(x-x_i)}{s-x}dx+\frac{h}{2}\sum_{i=0}^{m-1}\int_{x_i}^{x_{i+1}}\frac{u'(\xi_i)}{s-x}dx\\&\quad+\sum_{i=m+1}^{n-1}\int_{x_i}^{x_{i+1}}\frac{u'(\xi_i)(x-x_i)}{x-s}dx-\frac{h}{2}\sum_{i=m+1}^{n-1}\int_{x_i}^{x_{i+1}}\frac{u'(\xi_i)}{x-s}dx\\&=-\sum_{i=0}^{m-1}u'(\eta_i)\int_{x_i}^{x_{i+1}}\frac{x-x_i}{s-x}dx+\frac{h}{2}\sum_{i=0}^{m-1}u'(\zeta_i)\int_{x_i}^{x_{i+1}}\frac{1}{s-x}dx\\&\quad+\sum_{i=m+1}^{n-1}u'(\eta_i)\int_{x_i}^{x_{i+1}}\frac{x-x_i}{x-s}dx-\frac{h}{2}\sum_{i=m+1}^{n-1}u'(\zeta_i)\int_{x_i}^{x_{i+1}}\frac{1}{x-s}dx\\&=\sum_{i=0,i\neq m}^{n-1}u'(\eta_i)\int_{x_i}^{x_{i+1}}\frac{x-\hat{x}_i}{x-s}dx+\frac{h}{2}\sum_{i=0,i\neq m}^{n-1}[u'(\eta_i)-u'(\zeta_i)]\int_{x_i}^{x_{i+1}}\frac{1}{x-s}dx\\&=\sum_{i=0,i\neq m}^{n-1}[u'(\eta_i)-u'(s)]\mathcal{I}_{n,i}(s)+u'(s)\sum_{i=0,i\neq m}^{n-1}\mathcal{I}_{n,i}(s)+\frac{h}{2}\sum_{i=0,i\neq m}^{n-1}[u'(\eta_i)-u'(\zeta_i)]\ln\left|\frac{x_{i+1}-s}{x_i-s}\right|.\end{aligned} \tag{26}$$

Let

$$\mathcal{H}_m(x)=u(x)-u(\hat{x}_m)-u'(s)(x-\hat{x}_m)\quad x\in[x_m,x_{m+1}].$$

Then

$$⨍_{x_m}^{x_{m+1}}\frac{u(x)-u(\hat{x}_m)}{x-s}dx=⨍_{x_m}^{x_{m+1}}\frac{\mathcal{H}_m(x,s)}{x-s}dx+u'(s)\mathcal{I}_{n,m}(s). \tag{27}$$

Lemma 2 and (26), (27) imply

$$\mathcal{E}(s;u)=\mathcal{R}_1+\mathcal{R}_2+\mathcal{R}_3+\mathcal{R}_4+hu'(s)\ln\left(2\cos\frac{\tau\pi}{2}\right),$$

where

$$\mathcal{R}_1 = \fint_{x_m}^{x_{m+1}} \frac{\mathcal{H}_m(x)}{x-s}\,dx,$$
$$\mathcal{R}_2 = \sum_{i=0,i\neq m}^{n-1} [u'(\eta_i) - u'(s)]\mathcal{I}_{n,i}(s),$$
$$\mathcal{R}_3 = \frac{h}{2}\sum_{i=0,i\neq m}^{n-1} [u'(\eta_i) - u'(\zeta_i)]\ln\left|\frac{x_{i+1}-s}{x_i-s}\right|,$$
$$\mathcal{R}_4 = hu'(s)\left[\sum_{i=m+1}^{\infty} Q_1(2i+\tau) + \sum_{i=n-m}^{\infty} Q_1(-2i+\tau)\right].$$

Now we estimate $\mathcal{R}_i$ term by term. If $u \in C^{1+\alpha}[a,b]$ $(0 < \alpha \leqslant 1)$, we get

$$|\mathcal{H}_m^{(l)}(s)| \leqslant Ch^{1-l+\alpha} \quad l = 0, 1.$$

By the identity

$$\fint_a^b \frac{u(x)}{x-s}\,dx = u(s)\ln\frac{b-s}{s-a} + \int_a^b \frac{u(x)-u(s)}{x-s}\,dx,$$

we have

$$|\mathcal{R}_1| \leqslant \left|\mathcal{H}_m(s)\ln\frac{x_{m+1}-s}{s-x_m}\right| + \int_{x_m}^{x_{m+1}} \mathcal{H}'(\sigma(x))\,dx \leqslant C|\ln\gamma(h,s)|h^{1+\alpha}.$$

Here $\sigma(x) \in (x_m, x_{m+1})$. The second term $\mathcal{R}_2$ can be directly by Lemma 3. Since $u \in C^{1+\alpha}[a,b]$ $(0 < \alpha \leqslant 1)$, the definition of $\gamma(h,s)$ implies

$$|\mathcal{R}_3| \leqslant Ch^{1+\alpha}|\ln\gamma(h,s)|.$$

Finally, the last term $\mathcal{R}_4$ can be bounded by Lemma 2. Putting the above estimates together leads to the first two parts in (24). The last estimate in (24) can be obtained in a similar way by only noting the fact

$$|\mathcal{H}_m^{(l)}(s)| \leqslant Ch^{2-l+\alpha} \quad l = 0, 1$$

for $u(x) \in C^{2+\alpha}[a,b]$ $(0 < \alpha < 1)$. This concludes the proof. □

Corallory 1 (*Superconvergence*). *Let $\mathcal{Q}_h u(s)$ be computed by (4) and (5) with a uniform mesh, $0 < \alpha < 1$. Then at $s = x_m + (1+\tau^\star)h/2$ with $\tau^\star = \pm 2/3$, it holds*

$$|\mathcal{E}(s;u)| \leqslant \begin{cases} C[1+\eta(s)h^{1-\alpha}]h^{1+\alpha} & u(x) \in C^{1+\alpha}[a,b], \\ C[|\ln h| + \eta(s)]h^2 & u(x) \in C^2[a,b], \\ C\eta(s)h^2 & u(x) \in C^{2+\alpha}[a,b]. \end{cases} \tag{28}$$

3. A collocation scheme for Cauchy singular integral equation of the first kind

This section is devoted to the collocation scheme for the Cauchy singular integral equation (1). Using the midpoint rule approximate the singular integral in the left-hand side and collocating the resulting equation at two series of superconvergence points $\{s_i^k = x_i + (1+\tau_k)h/2\}|_{i=0}^{n-1}$ with $\tau_{1,2} = \pm 2/3$ respectively result in

$$\begin{cases} \mathcal{Q}_h u_h(s_i^1) = f(s_i^1), \\ \mathcal{Q}_h u_h(s_i^2) = f(s_i^2) \end{cases}$$

with $i = 0, 1, 2, \ldots, n-1$. To achieve some good properties of the coefficient matrix, we take the difference procedure for $\mathcal{Q}_h u_h(s_i^1)$ and $\mathcal{Q}_h u_h(s_i^2)$ at each subinterval and thus obtain the following computational scheme

$$\frac{\mathcal{Q}_h u_h(s_i^2) - \mathcal{Q}_h u_h(s_i^1)}{s_i^2 - s_i^1} = \frac{f(s_i^2) - f(s_i^1)}{s_i^2 - s_i^1} \quad i = 0, 1, \ldots, n-1.$$

That is,

$$\sum_{j=0}^{n-1}\left[\omega_j(s_i^2)-\omega_j(s_i^1)\right]u_j=f(s_i^2)-f(s_i^1)\quad i=0,1,\ldots,n-1, \tag{29}$$

which leads to a linear system

$$\mathcal{A}_n\mathbf{U}_a=\mathbf{F}_e, \tag{30}$$

where

$$\begin{aligned}&\mathcal{A}_n=(a_{ij})_{n\times n},\quad a_{ij}=\ln\left|\tfrac{36(j-i)^2-1}{36(j-i)^2-25}\right|,\\&\mathbf{U}_a=(u_0,u_1,\ldots,u_{n-1})^T,\quad \mathbf{F}_e=(\tilde{f}_0,\tilde{f}_1,\ldots,\tilde{f}_{n-1})^T\end{aligned} \tag{31}$$

with u_j the approximation of $u(x)$ at $\hat{x}_j$ and $\tilde{f}_i=f(s_i^2)-f(s_i^1)(i=0,1,2,\ldots,n-1)$. Finally, the approximation solution u_h can be computed by $u_h(x)=\sum_{j=0}^{n-1}u_j\varphi_j(x)$ with φ_j defined by (3).

Lemma 4. *Let $\mathcal{A}_n=(a_{ij})_{n\times n}$ be the matrix defined in (31). It holds*

(1) *$\mathcal{A}_n$ is a symmetric Toeplitz matrix and $-\mathcal{A}_n$ is M-matrix. Moreover, $\mathcal{A}_n$ is strictly diagonally dominant and thus, the linear system (30) has a unique solution.*
(2) *The eigenvalues of $\mathcal{A}_n$ satisfy that*

$$-C\leqslant\lambda(\mathcal{A}_n)\leqslant -Ch. \tag{32}$$

(3) *The condition number of $\mathcal{A}_n$ satisfies*

$$\mathrm{Cond}(\mathcal{A}_n)=\|\mathcal{A}_n\|_2\|\mathcal{A}_n^{-1}\|_2<Ch^{-1}. \tag{33}$$

Proof. Obviously, $\mathcal{A}_n$ is a symmetric Toeplitz matrix. Eq. (31) implies

$$a_{ii}=-\ln 25<0, \tag{34}$$

and otherwise,

$$a_{ij}=\ln\frac{36(j-i)^2-1}{36(j-i)^2-25}>0. \tag{35}$$

Therefore, $-\mathcal{A}_n$ is a Z-matrix. Moreover,

$$\begin{aligned}\sum_{j=0}^{n-1}a_{ij}&=\left(\sum_{j=0}^{i-1}+\sum_{j=i+1}^{n-1}\right)a_{ij}+a_{ii}=\left(\sum_{j=0}^{i-1}+\sum_{j=i+1}^{n-1}\right)\ln\frac{[6(j-i)-1][6(j-i)+1]}{[6(j-i)-5][6(j-i)+5]}-\ln 25\\&=\ln\frac{25(6i+1)[6(n-i)-5]}{(6i+5)[6(n-i)-1]}-\ln 25=\ln\frac{(6i+1)[6(n-i)-5]}{(6i+5)[6(n-i)-1]}<0.\end{aligned} \tag{36}$$

$\mathcal{A}_n$ is strictly diagonally dominant. Furthermore, by using the inequality

$$\ln(1+x)\geqslant Cx\quad x\in(0,1),$$

we obtain

$$\begin{aligned}-\sum_{j=0}^{n-1}a_{ij}&=\ln\frac{(6i+5)[6(n-i)-1]}{(6i+1)[6(n-i)-5]}=\ln\left(1+\frac{4}{6i+1}\right)+\ln\left[1+\frac{4}{6(n-i)-5}\right]\geqslant C\left[\frac{1}{6i+1}+\frac{1}{6(n-i)-5}\right]\\&\geqslant C[\eta(s_i^1)+\eta(s_i^2)]h.\end{aligned} \tag{37}$$

Gershgorin's circle theorem implies there exists some i, such that

$$|\lambda(\mathcal{A}_n)-a_{ii}|\leqslant\sum_{j=0,j\neq i}^{n-1}|a_{ij}|.$$

Combining (34)–(37) results in

$$|\lambda(\mathcal{A}_n)+\ln 25|\leqslant\sum_{j=0}^{n-1}a_{ij}+\ln 25\leqslant -C[\eta(s_i^1)+\eta(s_i^2)]h+\ln 25\leqslant -\frac{Ch}{b-a}+\ln 25,$$

which proves (32). Eq. (33) follows immediately from (32). Since an M-matrix is a Z-matrix with eigenvalues whose real parts are positive, we see that $-\mathcal{A}_n$ is an M-matrix. This concludes the proof. □

Theorem 2. *Assume that $u(x)$, the solution of the Cauchy singular integral equation (1), belongs to $C^{2+\alpha}[a,b]$ $(0 \leqslant \alpha \leqslant 1)$. Then, the solution of the linear system (30) satisfies*

$$\max_{0\leqslant i\leqslant n-1}|u(\hat{x}_i) - u_h(\hat{x}_i)| \leqslant Ch. \tag{38}$$

Proof. Let $\mathbf{U}_e$ be the exact solution vector, then

$$\mathcal{A}_n(\mathbf{U}_e - \mathbf{U}_a) = \mathcal{A}_n\mathbf{U}_e - \mathcal{F}_e,$$

which implies

$$\sum_{j=0}^{n-1} a_{ij}[u(\hat{x}_j) - u_j] = \mathcal{E}(s_i^2;u) - \mathcal{E}(s_i^1;u) \quad i = 0,1,\ldots,n-1. \tag{39}$$

It is straight forward that there exists some k, such that

$$u(\hat{x}_k) - u_k = \max_{0\leqslant i\leqslant n-1}|u(\hat{x}_i) - u_i|$$

or

$$u(\hat{x}_k) - u_k = -\max_{0\leqslant i\leqslant n-1}|u(\hat{x}_i) - u_i|.$$

We just prove the first case, since the second case can be proved similarly. Since $-\mathcal{A}_n$ is an M-matrix, from (39) we get

$$\begin{aligned}\mathcal{E}(s_k^2;u) - \mathcal{E}(s_k^1;u) &= \sum_{\substack{j=0\\ j\neq k}}^{n-1} a_{kj}[(u(\hat{x}_j) - u_j) - (u(\hat{x}_k) - u_k)] + (u(\hat{x}_k) - u_k)\sum_{j=0}^{n-1} a_{kj} \leqslant (u(\hat{x}_k) - u_k)\sum_{j=0}^{n-1} a_{ij}\\ &= (u(\hat{x}_k) - u_k)\ln\frac{(6k+1)[6(n-k)-5]}{(6k+5)[6(n-k)-1]},\end{aligned}$$

which leads to

$$\|u - u_h\|_\infty = u(\hat{x}_k) - u_k \leqslant \frac{1}{\ln\frac{(6k+5)[6(n-k)-1]}{(6k+1)[6(n-k)-5]}}[\mathcal{E}(s_k^1;u) - \mathcal{E}(s_k^2;u)].$$

Combining this with Corollary 1 and inequality (37), we have

$$\|u - u_h\|_\infty \leqslant C\frac{1}{[\eta(s_i^1) + \eta(s_i^2)]h}[|\mathcal{E}(s_i^1;u)| + |\mathcal{E}(s_i^2;u)|] \leqslant C\frac{1}{[\eta(s_i^1) + \eta(s_i^2)]h}[\eta(s_i^1) + \eta(s_i^2)]h^2 = Ch.$$

This concludes the proof. □

Remark 2. Before our superconvergence result, one may choose the middle points at each subinterval as the collocation points if the midpoint rule is applied. Such a natural choice results in a very simple collocation scheme

$$\mathcal{Q}_h u_h(\hat{x}_i) = f(\hat{x}_i),$$

i.e.,

$$\sum_{j=0}^{n-1}\omega_j(\hat{x}_i)u_j = f(\hat{x}_i) \quad i = 0,1,\ldots,n-1,$$

which leads to a linear system

$$\bar{\mathcal{A}}_n\mathbf{U}_a = \bar{\mathbf{F}}_e, \tag{40}$$

where

$$\begin{aligned}&\bar{\mathcal{A}}_n = (\bar{a}_{ij})_{n\times n}, \quad \bar{a}_{ij} = \ln\left|\frac{2(j-i)+1}{2(j-i)-1}\right|,\\ &\mathbf{U}_a = (u_0,u_1,\ldots,u_{n-1})^T, \quad \bar{\mathbf{F}}_e = (f(\hat{x}_0),f(\hat{x}_1),\ldots,f(\hat{x}_{n-1}))^T.\end{aligned} \tag{41}$$

It is worth mentioning that $\bar{\mathcal{A}}_n$ defined in (41) is a skew-symmetric Toeplitz matrix. Moreover, compared with the scheme (30), the accuracy of the scheme (40) seems to be $O(h^{1/2})$, which will be illustrated by an numerical experiment in Example 2.

4. Numerical examples

This section is devoted to some numerical examples to confirm our theoretical analysis given in the above sections.

Example 1. We consider the Cauchy principle value integral

$$\int_0^1 \frac{x^3}{x-s}\,dx \quad s \in (0,1). \tag{42}$$

The exact value is

$$\frac{1}{3}+\frac{s}{2}+s^2+s^3\ln\frac{1-s}{s}.$$

We use the rules $\mathcal{Q}_h u(s)$ and $\tilde{\mathcal{Q}}_h u(s)$ defined in (4) and (25) to evaluate (42) respectively. Numerical results are presented in Table 1 for $s = x_{[n/4]} + (1+\tau)h/2$ and Table 2 for $s = x_{n-1} + (1+\tau)h/2$. The errors are also graphically presented as log–log plots versus the mesh size h in Figs. 1 and 2. On the one hand, we can see from Fig. 1 that errors of the midpoint rule $\mathcal{Q}_h u(s)$ are $O(h^2)$ at superconvergence point ($\tau = \pm 2/3$) and $O(h)$ at non-superconvergence points, and errors of the modified midpoint rule $\tilde{\mathcal{Q}}_h u(s)$ are always $O(h^2)$ if the singular point s is far from two endpoints, which is in good agreement with our theoretical analysis. On the other hand, Fig. 2 shows that nearly first order accuracy for both rules can be explicitly observed

Table 1
Errors of the case where $s = x_{[n/4]} + (1+\tau)h/2$.

n	$\mathcal{Q}_h u(s)$			$\tilde{\mathcal{Q}}_h u(s)$		
	$\tau=-2/3$	$\tau=0$	$\tau=2/3$	$\tau=-2/3$	$\tau=0$	$\tau=2/3$
64	6.39077e−05	2.16431e−03	5.67612e−05	6.39077e−05	4.70187e−06	5.67612e−05
128	1.58304e−05	1.04836e−03	1.40458e−05	1.58304e−05	1.03403e−06	1.40458e−05
256	3.93931e−06	5.15881e−04	3.49265e−06	3.93931e−06	2.41090e−07	3.49265e−06
512	9.82544e−07	2.55883e−04	8.70774e−07	9.82544e−07	5.81118e−08	8.70774e−07
1024	2.45351e−07	1.27430e−04	2.17394e−07	2.45351e−07	1.42589e−08	2.17394e−07

Table 2
Errors of the case where $s = x_{n-1} + (1+\tau)h/2$.

n	$\mathcal{Q}_h u(s)$			$\tilde{\mathcal{Q}}_h u(s)$		
	$\tau=-2/3$	$\tau=0$	$\tau=2/3$	$\tau=-2/3$	$\tau=0$	$\tau=2/3$
64	4.95513e−03	3.94256e−02	1.73385e−02	4.95513e−03	7.44005e−03	1.73385e−02
128	2.37884e−03	1.97878e−02	8.64842e−03	2.37884e−03	3.66884e−03	8.64842e−03
256	1.16179e−03	9.91000e−03	4.31636e−03	1.16179e−03	1.81888e−03	4.31636e−03
512	5.73289e−04	4.95836e−03	2.15556e−03	5.73289e−04	9.04878e−04	2.15556e−03
1024	2.84572e−04	2.47985e−03	1.07696e−03	2.84572e−04	4.51131e−04	1.07696e−03

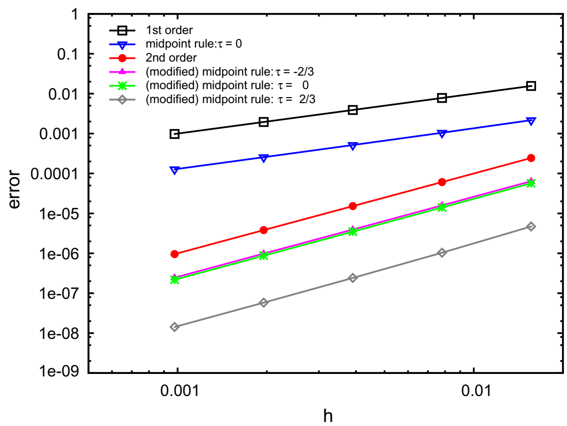

Fig. 1. Errors of $\mathcal{Q}_h u(s)$and $\tilde{\mathcal{Q}}_h u(s)$ where $s = x_{[n/4]} + (1+\tau)h/2$.

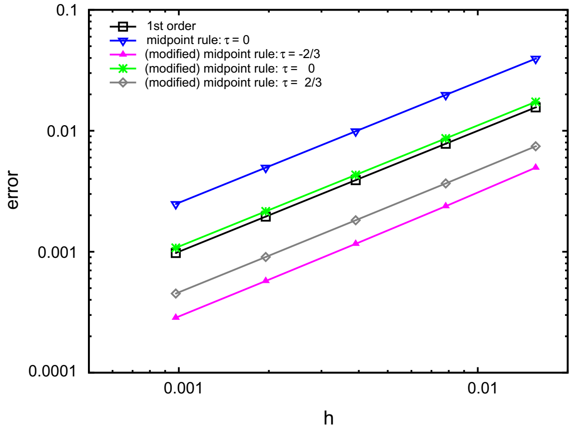

Fig. 2. Errors of $\mathcal{Q}_h u(s)$and $\tilde{\mathcal{Q}}_h u(s)$ where $s = x_{n-1} + (1+\tau)h/2$.

if the singular point s is very close to the endpoints. In such case, no superconvergence phenomenon occurs since $\eta(s)$ in (13) is proportional to h^{-1}.

Example 2. Consider the Cauchy singular integral equation

$$\int_{-1}^{1} \frac{u(x)}{x-s}\,dx = \frac{4}{3} - 2s^2 + (s^3 - s)\ln\frac{1+s}{1-s}.$$

The exact solution is $u(x) = -x^3 + x$. We examine the maximal nodal errors and condition numbers of two schemes discussed in Section 3. Numerical results are presented in Table 3, and also graphically presented as log–log plots versus the mesh size h in Figs. 3 and 4. From Fig. 3, we see that the condition numbers of both schemes increases as n with the order $O(n)$, the same as the estimate in Lemma 4. Fig. 4 indicates that the maximal nodal errors of the scheme (30) can reach $O(h)$ while

Table 3
Numerical results for Example 2.

n	Scheme (30)		Scheme (40)	
	$\|u-u_h\|_\infty$	Cond($\mathcal{A}_n$)	$\|u-u_h\|_\infty$	Cond($\bar{\mathcal{A}}_n$)
64	5.58385e−03	69.2192	6.53214e−02	54.8054
128	3.03724e−03	138.6631	4.41462e−02	109.7400
256	1.60477e−03	277.6250	3.00650e−02	219.5413
512	8.32499e−04	555.6236	2.06503e−02	439.0907
1024	4.26814e−04	1111.696	1.42898e−02	878.1451
2048	2.17637e−04	2223.915	9.94696e−03	1756.214

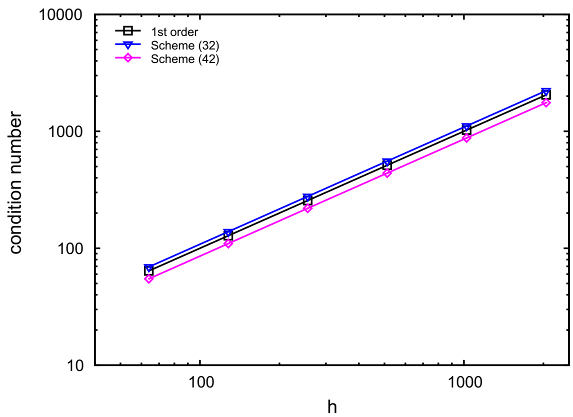

Fig. 3. Condition numbers of schemes (30) and (40).

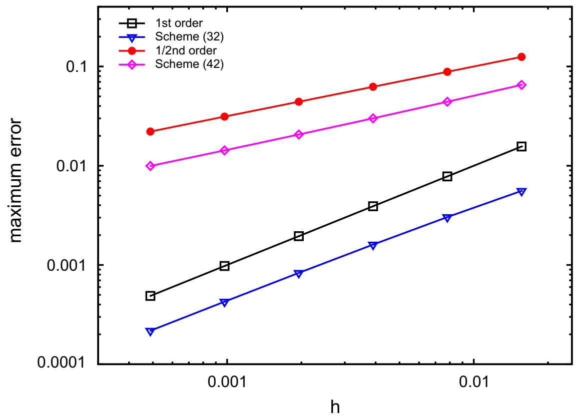

Fig. 4. Errors of schemes (30) and (40).

that of the scheme (40) seems only $O(h^{1/2})$. Through this example, we know that under the case of almost the same computational cost, the scheme (30) based on the superconvergence results has superiority over the scheme (40).

5. Concluding remarks

We have shown the superconvergence estimate of the composite midpoint rule for the Cauchy principal value integral in the left-hand side of (1) and also suggested a modified midpoint rule, whose global accuracy can reach $O(h^2)$, one order higher than that of the original one. From the superconvergence analysis, we see that there exists two superconvergence points at each subinterval which is not closed to two endpoints. Thus, a natural idea raised is that whether we can use this result to solve the Cauchy singular integral equation (1). In the last half of the paper, an efficient collocation scheme (30) has been proposed by making fully use of informations of all superconvergence points. The coefficient matrix of its resulting system possesses many good properties, such as symmetric Toeplitz type, M-type and diagonally dominant, from which we obtained an optimal error estimate.

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References

[1] M.A. Abdou, A.A. Nasr, On the numerical treatment of the singular integral equation of the second kind, Appl. Math. Comput. 146 (2003) 373–380.
[2] S. Amari, Evaluation of Cauchy principal-value integrals using modified Simpson rules, Appl. Math. Lett. 7 (1994) 19–23.
[3] L.C. Andrews, Special Functions of Mathematics for Engineers, second ed., McGraw-Hill, Inc., 1992.
[4] M. Bonis, C. Laurita, Numerical solution of systems of Cauchy singular integral equations with constant coefficients, Appl. Math. Comput. 219 (2012) 1391–1410.
[5] A. Chakrabarti, V.G. Berghe, Approximation solution of singular integral equations, Appl. Math. Lett. 17 (2004) 553–559.
[6] G.A. Chandler, Midpoint collocation for Cauchy singular integral equations, Numer. Math. 62 (1992) 483–509.
[7] U.J. Choi, S.W. Kim, B.I. Yun, Improvement of the asymptotic behaviour of the Euler–Maclaurin formula for Cauchy principal value and Hadamard finite-part integrals, Int. J. Numer. Meth. Engng. 61 (2004) 496–513.
[8] C. Dagnino, E.S. L'Aquila, On the evaluation of one-dimensional Cauchy principal value integrals by rules based on cubic spline interpolation, Computing 43 (1990) 267–276.
[9] K. Diethelm, Modified compound quadrature rules for strongly singular integrals, Computing 52 (1994) 337–354.
[10] K. Diethelm, Gaussian quadrature formulae of the third kind for Cauchy principal value integrals: basic properties and error estimates, J. Comput. Appl. Math. 65 (1995) 97–114.
[11] K. Diethelm, Asymptotically sharp error bounds for a quadrature rule for Cauchy principal value integrals based on piecewise linear interpolation, Approx. Theory Its Appl. 11 (1995) 78–89.
[12] Z.K. Eshkuvatov, N.M.A. Nik Long, M. Abdulkawi, Quadrature formula for approximating the singular of Cauchy type with unbounded weight function on the edges, J. Comput. Appl. Math. 233 (2009) 334–345.
[13] T. Hasegawa, Uniform approximations to finite Hilbert transform and its derivative, J. Comput. Appl. Math. 163 (2004) 127–138.
[14] N.I. Ioakimidis, On the uniform convergence of Gaussian quadrature rules for Cauchy principal value integrals and their derivatives, Math. Comp. 44 (1985) 191–198.
[15] N.I. Ioakimidis, The successive approximations method for the airfoil equation, J. Comput. Appl. Math. 21 (1988) 231–238.
[16] S. Kim, Solving the singular integral equation using Gaussian quadrature and overdetermined system, Appl. Math. Comput. 35 (1998) 63–71.

[17] P. Köhler, On the error of quadrature formulae for Cauchy principal value integrals based on piecewise interpolation, Approx. Theory Its Appl. 13 (1997) 58–67.
[18] B.Y. Li, W. Sun, Newton Cotes rules for Hadamard finite-part integrals on an interval, IMA J. Numer. Anal. 30 (2009) 1235–1255.
[19] Hongwei Li, Kean Chen, Gauss quadrature rules for Cauchy principal value integrals with wavelets, Appl. Math. Comput. 186 (2007) 357–364.
[20] I.K. Lifanov, L.N. Poltavskii, G.M. Vainikko, Hypersingular Integral Equations and Their Applications, CRC Press, 2004.
[21] P. Linz, On the approximate computation of certain strongly singular integrals, Computing 35 (1985) 345–353.
[22] D.J. Liu, J.M. Wu, D.H. Yu, The superconvergence of Newton–Cotes rule for Cauchy principal integrals, J. Comput. Appl. Math. 235 (2010) 696–707.
[23] A. Mennouni, S. Guedjiba, A note on solving Cauchy integral equations of the first kind by iterations, Appl. Math. Comput. 217 (2011) 7442–7447.
[24] H.N. Mülthei, C. Schneider, Discrete collocation for a first kind Cauchy singular integral equation with weakly singular solution, J. Integr. Eq. Appl. 9 (1997) 341–377.
[25] A.P. Orsi, Spline approximation for Cauchy principal value integrals, J. Comput. Appl. Math. 30 (1990) 191–201.
[26] J.M. Wu, Y. Lü, A superconvergence result for the second-order Newton-Cotes formula for certain finite-part integrals, IMA J. Numer. Anal. 25 (2005) 253–263.
[27] J.M. Wu, W. Sun, The superconvergence of the composite trapezoidal rule for Hadamard finite part integrals, Numer. Math. 102 (2005) 343–363.
[28] J.M. Wu, W. Sun, The superconvergence of Newton–Cotes rules for the Hadamard finite-part integral on an interval, Numer. Math. 109 (2008) 143–165.
[29] D.H. Yu, Natural Boundary Integral Method and Its Applications, Kluwer Academic Publishers., 2002.
[30] X.P. Zhang, J.M. Wu, D.H. Yu, Superconvergence of the composite Simpson's rule for certain finite-part integral and its applications, J. Comp. Appl. Math. 223 (2009) 598–613.