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Numerical solution of a certain hypersingular integral equation of the first kind

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Abstract In this paper, we first discuss the midpoint rule for evaluating hypersingular integrals with the kernel $\sin^{-2}[(x-s)/2]$ defined on a circle, and the key point is placed on its *pointwise superconvergence phenomenon*. We show that this phenomenon occurs when the singular point s is located at the midpoint of each subinterval and obtain the corresponding supercovergence analysis. Then we apply the rule to construct a collocation scheme for solving the relevant hypersingular integral equation, by choosing the midpoints as the collocation points. It's interesting that the inverse of coefficient matrix for the resulting linear system has an explicit expression, by which an optimal error estimate is established. At last, some numerical experiments are presented to confirm the theoretical analysis.

Keywords Hypersingular integral · Hypersingular integral equation · Midpoint rule · Superconvergence · Collocation

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1 Introduction

Consider the hypersingular integral defined on a circle

$$I(s,u) := \int\!\!\!\!\!\!=_0^{2\pi} \frac{u(x)}{\sin^2 \frac{x-s}{2}}\,dx, \quad s \in (0, 2\pi), \tag{1.1}$$

where s is the singular point. Hypersingular integral (1.1) must be understood in the Hadamard sense and it possesses several definitions equivalent mathematically. Here we adopt the definition as follows:

$$\int\!\!\!\!\!\!=_0^{2\pi} \frac{u(x)}{\sin^2 \frac{x-s}{2}}\,dx = \lim_{\varepsilon\to 0}\left\{\left(\int_0^{s-\varepsilon} + \int_{s+\varepsilon}^{2\pi}\right)\frac{u(x)}{\sin^2 \frac{x-s}{2}}\,dx - \frac{8u(s)}{\varepsilon}\right\} \tag{1.2}$$

The density function $u(x)$ is said to be finite-part integrable with respect to the weight $\sin^{-2}[(x-s)/2]$ if the limit on the right-hand side of (1.2) exists. A sufficient condition for $u(x)$ to be finite-part integrable is that its first derivative $u'(x)$ is Hölder continuous.

Such integrals are frequently encountered in many physical problems, such as in fracture mechanics, elasticity problems, acoustics, aerodynamics as well as electromagnetic scattering [14, 17, 23], and in boundary element methods (BEMs) [5, 6, 13, 15, 21, 30, 35]. For example, for some boundary value problems in unbounded domain, by introducing a circle or an ellipse as an artificial boundary and using Green formula, one can obtain the corresponding hypersingular integral equation on this artificial boundary. This method is often known as "Dirichlet to Neumann method (DtN method)".

In recent decades, there have been a lot of works in developing efficient quadrature methods for hypersingular integrals on an interval

$$\int\!\!\!\!\!\!=_a^{b} \frac{u(x)}{(x-s)^{p+1}}\,dx = \lim_{\varepsilon\to 0}\left\{\left(\int_a^{s-\varepsilon} + \int_{s+\varepsilon}^{b}\right)\frac{u(x)}{(x-s)^{p+1}}\,dx - \frac{2u^{(p-1)}(s)}{\varepsilon}\right\},$$
$$s \in (a,b),\ p = 1, 2, \tag{1.3}$$

such as Gaussian method [10, 11, 20, 22, 25, 26], Newton-Cotes method [24, 27–29, 31, 36–38], transformation method [2, 7, 8] and some others [4, 9, 12]. Relatively speaking, the quadrature method for (1.1) has less been studied, references [7, 8, 14, 34, 37, 38] may in fact be the entire literatures on the subject. For the integrals with periodic kernels, a well technique is the Euler-Maclarin expansion. In [7, 8], Elliott and Venturino extended the results by Lyness [19] to hypersingular integrals by differentiating the Lyness' formula. Kress [14] described a fully discrete method based on trigonometric interpolation for the numerical solution of the hypersingular integral equation arising from scattering problem. This method is also applicable for evaluating (1.1) and can produce the exponential accuracy if $u(x)$ is sufficiently smooth. However, if $u(x)$ has a lower regularity, its accuracy will descend. At this time, Newton-Cotes method becomes competitive.

Newton-Cotes method for evaluating (1.1) was firstly discussed in [34], whose idea originated from [18], where the composite trapezoidal rule is suggested and an accuracy of $O(h)$ is obtained when the singular point s is located at the midpoint of some subinterval. In fact, Newton-Cotes method can reach a higher-order convergence rate when the singular point s coincides with some a priori known point. This is the so-called *pointwise superconvergence phenomenon* of the Newton-Cotes method for hypersingular integrals. The superconvergence of Newton-Cotes method was studied in [16, 27–29, 36] for (1.3) and in [37, 38] for (1.1).

The composite midpoint rule, one of the lowest order Newton-Cotes rules, is widely used in the evaluation of integrals with smooth, weakly or Cauchy singular kernels due to its simplicity and ease of implementation. However, this rule is divergent in general for hypersingular integrals, which may explain why it has drawn so little attention. In this paper, we show that, in despite of the global divergence, this rule is still valid in the evaluation of (1.1). The key point is placed on the investigation of its superconvergence. Furthermore, we apply this superconvergence result to construct a collocation scheme for solving the relevant hypersingular integral equation, and obtain an optimal error estimate.

The rest of this paper is organized as follows. In Sect. 2, we introduce the midpoint rule for (1.1) by employing the piecewise constant interpolation. In Sect. 3, the superconvergence result of the rule is established and a modified rule is suggested. In Sect. 4, we present a collocation scheme for solving a certain hypersingular integral equation. Based on the superconvergence result, an optimal error estimate is established. Finally, several numerical examples are illustrated to validate our analysis.

2 The composite midpoint rule

Let $0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 2\pi$ be a uniform partition of the interval $[0, 2\pi]$ with the mesh size $h = 2\pi/n$. Let $u_0^I(x)$ be the piecewise constant interpolant of $u(x)$, defined by

$$u_0^I(x) = \sum_{i=1}^{n} u(\hat{x}_i)\varphi_i(x),$$

where $\hat{x}_i = (x_{i-1} + x_i)/2$, and

$$\varphi_i(x) = \begin{cases} 1, & \text{on } [x_{i-1}, x_i], \\ 0, & \text{otherwise.} \end{cases}$$

Replacing $u(x)$ in (1.1) with $u_0^I(x)$, we obtain the composite midpoint rule

$$\mathscr{Q}_{0n}(s, u) := \neq\!\!\!\!\!\!\int_0^{2\pi} \frac{u_0^I(x)}{\sin^2 \frac{x-s}{2}}\, dx = \sum_{i=1}^{n} \omega_i^0(s) u(\hat{x}_i) = \neq\!\!\!\!\!\!\int_0^{2\pi} \frac{u(x)}{\sin^2 \frac{x-s}{2}}\, dx - \mathscr{E}_{0n}(s, u), \tag{2.1}$$

where $\omega_i^0(s)$ denote the Cotes coefficients, and $\mathscr{E}_{0n}(s,u)$ the error functional. By (1.2) and through direct calculations, we have

$$\omega_i^0(s) = 2\left(\cot\frac{x_{i-1}-s}{2} - \cot\frac{x_i-s}{2}\right). \tag{2.2}$$

Throughout the paper, C will denote a generic constant which is independent of h and s and it may have different values in different places. In addition, we assume $u(x)$ is 2π-periodic and $s \in (x_{m-1}, x_m)$ for some m. Let $s = x_{m-1} + (\tau+1)h/2$ with $\tau \in (-1,1)$ denoting its local coordinate. Define $\gamma(h,s) = \min_{0\le i\le n}|s-x_i|/h$, i.e.,

$$\gamma(h,s) = \begin{cases} \frac{1+\tau}{2}, & \tau \le 0,\\ \frac{1-\tau}{2}, & \tau > 0, \end{cases} \tag{2.3}$$

$$\kappa_s(x) = \begin{cases} \frac{(x-s)^2}{\sin^2\frac{x-s}{2}}, & x \ne s,\\ 4, & x = s, \end{cases} \tag{2.4}$$

$$\,\,\int\!\!\!\!\!\!=_{x_{m-1}}^{x_m} \frac{u(x)}{\sin^2\frac{x-s}{2}}\,dx = \lim_{\varepsilon\to 0}\left\{\left(\int_{x_{m-1}}^{s-\varepsilon} + \int_{s+\varepsilon}^{x_m}\right)\frac{u(x)}{\sin^2\frac{x-s}{2}}\,dx - \frac{8u(s)}{\varepsilon}\right\}$$

and

$$\mathscr{I}_{n,i}(s) = \begin{cases} \int_{x_{i-1}}^{x_i} \frac{x-\hat{x}_i}{\sin^2\frac{x-s}{2}}\,dx, & i \ne m,\\ \,\,\int\!\!\!\!\!\!=_{x_{m-1}}^{x_m} \frac{x-\hat{x}_m}{\sin^2\frac{x-s}{2}}\,dx, & i = m. \end{cases} \tag{2.5}$$

Lemma 2.1 *Assume $s = x_{m-1} + (\tau+1)h/2$ with $\tau \in (-1,1)$. Let $\mathscr{I}_{n,i}(s)$ be defined in (2.5), we have*

$$\begin{aligned}\mathscr{I}_{n,i}(s) = &-2h\sum_{k=1}^{\infty}\{\sin[k(x_i-s)] + \sin[k(x_{i-1}-s)]\}\\ &-4\sum_{k=1}^{\infty}\frac{1}{k}\{\cos[k(x_i-s)] - \cos[k(x_{i-1}-s)]\}.\end{aligned} \tag{2.6}$$

Proof For $i = m$,

$$\begin{aligned}\mathscr{I}_{n,m}(s) &= \lim_{\varepsilon\to 0}\left\{\left(\int_{x_{m-1}}^{s-\varepsilon} + \int_{s+\varepsilon}^{x_m}\right)\frac{x-\hat{x}_m}{\sin^2\frac{x-s}{2}}\,dx - \frac{8(s-\hat{x}_m)}{\varepsilon}\right\}\\ &= -h\cot\frac{x_m-s}{2} - h\cot\frac{x_{m-1}-s}{2}\\ &\quad + 2\lim_{\varepsilon\to 0}\left\{\left(\int_{x_{m-1}}^{s-\varepsilon} + \int_{s+\varepsilon}^{x_m}\right)\cot\frac{x-s}{2}\,dx\right\}.\end{aligned} \tag{2.7}$$

Similarly, for $i \neq m$, using integral by parts on the corresponding Riemann integral, we have

$$\mathscr{I}_{n,i}(s) = -h \cot \frac{x_i - s}{2} - h \cot \frac{x_{i-1} - s}{2} + 2 \int_{x_{i-1}}^{x_i} \cot \frac{x - s}{2}\, dx. \qquad (2.8)$$

Now, by using the well-known identity (see [35])

$$\frac{1}{2} \cot \frac{t}{2} = \sum_{k=1}^{\infty} \sin kt,$$

we can easily obtain (2.6) from (2.7) and (2.8). The proof is completed. □

Lemma 2.2 *Under the same assumptions of Lemma 2.1, we have*

$$\sum_{i=1}^{n} \mathscr{I}_{n,i}(s) = -4\pi \tan \frac{\tau\pi}{2}.$$

Proof By (2.6), we have

$$\begin{aligned}
\sum_{i=1}^{n} \mathscr{I}_{n,i}(s) &= -2h \sum_{k=1}^{\infty} \sum_{i=1}^{n} \{\sin[k(x_i - s)] + \sin[k(x_{i-1} - s)]\} \\
&\quad - 4 \sum_{k=1}^{\infty} \frac{1}{k} \sum_{i=1}^{n} \{\cos[k(x_i - s)] - \cos[k(x_{i-1} - s)]\} \\
&= -4h \sum_{k=1}^{\infty} \sum_{i=1}^{n} \sin[k(x_i - s)] = -4h \sum_{j=1}^{\infty} n \sin[nj(x_1 - s)] \\
&= 8\pi \sum_{j=1}^{\infty} \sin[j(1+\tau)\pi] = 4\pi \cot \frac{(1+\tau)\pi}{2} = -4\pi \tan \frac{\tau\pi}{2},
\end{aligned}$$

where

$$\sum_{i=1}^{n} \sin[k(x_i - s)] = \begin{cases} n \sin[k(x_1 - s)], & k = nj, \\ 0, & \text{otherwise} \end{cases}$$

has been used. The proof is completed. □

Remark 2.1 From (2.5) and Lemma 2.2, we can see that in the special case $u(x) = x$, the error term $\mathscr{E}_{0n}(s, x) = -4\pi \tan(\tau\pi)/2$ does not vanish in general. Hence the composite midpoint rule cannot be directly used in practical computation before some special treatment.

3 The superconvergence and a modified midpoint rule

In the above section, we have shown that the composite midpoint rule is generally useless by an interesting example in Remark 2.1. In order to build it into a useful one, we suggest two approaches: one is using the superconvergence property of the composite midpoint rule, another is modifying the composite midpoint rule slightly to get a new one. Before stating the main theorem of this section, we need to introduce another lemma.

Define

$$\bar{\mathscr{R}}(s) = \begin{cases} \sum_{\substack{i=1, \\ i\neq m-1,m}}^{n} [u'(\eta_i) - u'(s)]\mathscr{I}_{n,i}(s), & -1 < \tau \le 0, \\ \sum_{\substack{i=1, \\ i\neq m,m+1}}^{n} [u'(\eta_i) - u'(s)]\mathscr{I}_{n,i}(s), & 0 < \tau < 1, \end{cases} \tag{3.1}$$

where $\eta_i \in [x_{i-1}, x_i]$.

Lemma 3.1 *Under the same assumption of Lemma 2.1, we have*

$$\bar{\mathscr{R}}(s) \le \begin{cases} C\rho(s)h^{1+\alpha}, & u(x) \in C^{2+\alpha}[0, 2\pi], \\ C\rho(s)h^{2}|\ln h|, & u(x) \in C^{3}[0, 2\pi], \\ C\rho(s)h^{2}, & u(x) \in C^{3+\alpha}[0, 2\pi], \end{cases} \tag{3.2}$$

where $0 < \alpha < 1$ *and*

$$\rho(s) = \max_{0\le x\le 2\pi} \{\kappa_s^{3/2}(x)\}. \tag{3.3}$$

Here we call $u(x) \in C^{k+\alpha}[0, 2\pi]$, $(0 \le \alpha \le 1$ *and* k *is an integer number*), *if* $u(x) \in C^{k}[0, 2\pi]$ *and*

$$|u^{(k)}(\xi) - u^{(k)}(\eta)| \le C|\xi - \eta|^{\alpha}, \quad \forall \xi, \eta \in [0, 2\pi].$$

Proof We only consider the case $-1 < \tau \le 0$, since the proof of the case $0 < \tau < 1$ can be obtained similarly. From (2.8), $\mathscr{I}_{n,i}(s)$, $(i \ne m-1, m)$ is actually the error of the trapezoidal rule for certain Riemann integral on $[x_{i-1}, x_i]$. Thus there exists $\tilde{x}_i \in (x_{i-1}, x_i)$, such that

$$\mathscr{I}_{n,i}(s) = \frac{h^3 \cos\frac{\tilde{x}_i - s}{2}}{12\sin^3\frac{\tilde{x}_i - s}{2}}, \quad i \ne m-1, m.$$

If $u(x) \in C^{2+\alpha}[0, 2\pi]$, $(0 < \alpha \le 1)$, we have

$$\left| \sum_{\substack{i=1, \\ i\neq m-1,m}}^{n} [u'(\eta_i) - u'(s)]\mathscr{I}_{n,i}(s) \right|$$
$$\le \sum_{\substack{i=1, \\ i\neq m-1,m}}^{n} \frac{h^3|\eta_i - s|^{1+\alpha}}{12|\tilde{x}_i - s|^3} \frac{|\tilde{x}_i - s|^3}{|\sin^3\frac{\tilde{x}_i - s}{2}|} \left|\cos\frac{\tilde{x}_i - s}{2}\right|$$

$$\leq C \max_{0\leq x\leq 2\pi}\{\kappa_s^{3/2}(x)\}\left[\sum_{i=1}^{m-2}\frac{h^3(s-x_{i-1})^{1+\alpha}}{12(s-x_i)^3}+\sum_{i=m+1}^{n}\frac{h^3(x_i-s)^{1+\alpha}}{12(x_{i-1}-s)^3}\right], \tag{3.4}$$

where $\kappa_s(x)$ is defined in (2.4). Noting that $s = x_{m-1} + (\tau + 1)h/2$, $(-1 < \tau \leq 0)$, we have

$$\begin{aligned}\sum_{i=1}^{m-2}\frac{h^3(s-x_{i-1})^{1+\alpha}}{12(s-x_i)^3} &\leq \sum_{i=1}^{m-2}\frac{h^{4+\alpha}+h^3(s-x_i)^{1+\alpha}}{12(s-x_i)^3}\\ &\leq \frac{h^{1+\alpha}}{12}\sum_{i=1}^{m-2}\frac{1+(m-1+\frac{1+\tau}{2}-i)^{1+\alpha}}{(m-1+\frac{1+\tau}{2}-i)^3}\\ &\leq \begin{cases} Ch^{1+\alpha}, & 0<\alpha<1,\\ Ch^2|\ln h|, & \alpha=1\end{cases}\end{aligned} \tag{3.5}$$

and similarly,

$$\sum_{i=m+1}^{n}\frac{h^3(x_i-s)^{1+\alpha}}{12(x_{i-1}-s)^3} \leq \begin{cases} Ch^{1+\alpha}, & 0<\alpha<1,\\ Ch^2|\ln h|, & \alpha=1.\end{cases} \tag{3.6}$$

Thus, the first two bounds in (3.2) can be directly obtained from (3.4), (3.5) and (3.6). If $u(x) \in C^{3+\alpha}[0, 2\pi]$, $(0 < \alpha < 1)$, by a similar approach, we get

$$\begin{aligned}&\left|\sum_{\substack{i=1,\\ i\neq m-1,m}}^{n}[u'(\eta_i)-u'(s)]\mathscr{I}_{n,i}(s)\right|\\ &\quad\leq C \max_{0\leq x\leq 2\pi}\{\kappa_s^{3/2}(x)\}\left[\sum_{i=1}^{m-2}\frac{h^3(s-x_{i-1})^{2+\alpha}}{12(s-x_i)^3}+\sum_{i=m+1}^{n}\frac{h^3(x_i-s)^{2+\alpha}}{12(x_{i-1}-s)^3}\right],\end{aligned} \tag{3.7}$$

$$\begin{aligned}\sum_{i=1}^{m-2}\frac{h^3(s-x_{i-1})^{2+\alpha}}{12(s-x_i)^3} &\leq \frac{h^{2+\alpha}}{12}\sum_{i=1}^{m-2}\frac{1+(m-1+\frac{1+\tau}{2}-i)^{2+\alpha}}{(m-1+\frac{1+\tau}{2}-i)^3}\\ &\leq \frac{h^{2+\alpha}}{12}\sum_{i=1}^{m-2}\left\{\frac{1}{(m-1+\frac{1+\tau}{2}-i)^3}+\frac{1}{i^{1-\alpha}}\right\}\leq Ch^2,\end{aligned} \tag{3.8}$$

and analogously,

$$\sum_{i=m+1}^{n}\frac{h^3(x_i-s)^{2+\alpha}}{12(x_{i-1}-s)^3} \leq Ch^2. \tag{3.9}$$

Putting (3.7), (3.8) and (3.9) together yields the last bound in (3.2). The proof is completed. □

Now, we state the main result of this section in the following theorem.

Theorem 3.1 *Let $\mathscr{Q}_{0n}(s,u)$ be computed by (2.1) and (2.2) with a uniform mesh. Then we have*

$$I(s,u) - \mathscr{Q}_{0n}(s,u) = -4\pi u'(s)\tan\frac{\tau\pi}{2} + \mathscr{R}(s), \tag{3.10}$$

where

$$|\mathscr{R}(s)| \le \begin{cases} C\rho(s)\gamma^{-1}(h,s)h^{1+\alpha}, & u(x)\in C^{2+\alpha}[0,2\pi], \\ C\rho(s)\gamma^{-1}(h,s)h^{2}|\ln h|, & u(x)\in C^{3}[0,2\pi], \\ C\rho(s)\gamma^{-1}(h,s)h^{2}, & u(x)\in C^{3+\alpha}[0,2\pi] \end{cases} \tag{3.11}$$

with $0<\alpha<1$ and $\rho(s)$ is defined in (3.3).

Proof We only need to consider the case $-1<\tau\le 0$, since the case $0<\tau<1$ can be obtained by the same approach. By the property of Taylor expansion, there exists $\xi_i\in(x_{i-1},x_i)$, such that

$$u(x)-u(\hat{x}_i) = u'(\xi_i)(x-\hat{x}_i), \quad x\in[x_{i-1},x_i],\ i\ne m-1,m.$$

Then, by the means value theorem of integration and Lemma 2.2, we have

$$\begin{aligned}
&\left(\int_c^{x_{m-2}} + \int_{x_m}^{c+2\pi}\right)\frac{u(x)-u(\hat{x}_i)}{\sin^2\frac{x-s}{2}}\,dx \\
&= \sum_{\substack{i=1,\\ i\ne m-1,m}}^{n}\int_{x_{i-1}}^{x_i}\frac{u'(\xi_i)(x-\hat{x}_i)}{\sin^2\frac{x-s}{2}}\,dx \\
&= \sum_{\substack{i=1,\\ i\ne m-1,m}}^{n}\int_{x_{i-1}}^{x_i}\frac{u'(\xi_i)(x-x_{i-1})}{\sin^2\frac{x-s}{2}}\,dx - \frac{h}{2}\sum_{\substack{i=1,\\ i\ne m-1,m}}^{n}\int_{x_{i-1}}^{x_i}\frac{u'(\xi_i)}{\sin^2\frac{x-s}{2}}\,dx \\
&= \sum_{\substack{i=1,\\ i\ne m-1,m}}^{n}u'(\eta_i)\int_{x_{i-1}}^{x_i}\frac{(x-x_{i-1})}{\sin^2\frac{x-s}{2}}\,dx - \frac{h}{2}\sum_{\substack{i=1,\\ i\ne m-1,m}}^{n}u'(\zeta_i)\int_{x_{i-1}}^{x_i}\frac{1}{\sin^2\frac{x-s}{2}}\,dx \\
&= \sum_{\substack{i=1,\\ i\ne m-1,m}}^{n}u'(\eta_i)\mathscr{I}_{n,i}(s) + \frac{h}{2}\sum_{\substack{i=1,\\ i\ne m-1,m}}^{n}[u'(\eta_i)-u'(\zeta_i)]\int_{x_{i-1}}^{x_i}\frac{1}{\sin^2\frac{x-s}{2}}\,dx \\
&= \sum_{\substack{i=1,\\ i\ne m-1,m}}^{n}[u'(\eta_i)-u'(s)]\mathscr{I}_{n,i}(s) \\
&\quad - u'(s)\mathscr{I}_{n,m-1}(s) - u'(s)\mathscr{I}_{n,m}(s) - 4\pi u'(s)\tan\frac{\tau\pi}{2} \\
&\quad + h\sum_{\substack{i=1,\\ i\ne m-1,m}}^{n}\big[u'(\zeta_i)-u'(\eta_i)\big]\left[\cot\frac{x_i-s}{2} - \cot\frac{x_{i-1}-s}{2}\right],
\end{aligned} \tag{3.12}$$

where $\eta_i, \zeta_i \in [x_{i-1}, x_i]$. Setting

$$\mathscr{H}_i(x) = u(x) - u(\hat{x}_i) - u'(s)(x - \hat{x}_i), \quad x \in [x_{i-1}, x_i],\ i = m-1, m,$$

then

$$\begin{aligned}
\fint_{x_{m-1}}^{x_m} \frac{u(x) - u(\hat{x}_m)}{\sin^2 \frac{x-s}{2}}\, dx &= \fint_{x_{m-1}}^{x_m} \frac{\mathscr{H}_m(x)}{\sin^2 \frac{x-s}{2}}\, dx + u'(s)\mathscr{I}_{n,m}(s) \\
&= 4 \fint_{x_{m-1}}^{x_m} \frac{\mathscr{H}_m(x)}{(x-s)^2}\, dx \\
&\quad + \fint_{x_{m-1}}^{x_m} \frac{\mathscr{H}_m(x)[\kappa_s(x) - 4]}{(x-s)^2}\, dx + u'(s)\mathscr{I}_{n,m}(s)
\end{aligned} \tag{3.13}$$

and

$$\int_{x_{m-2}}^{x_{m-1}} \frac{u(x) - u(\hat{x}_{m-1})}{\sin^2 \frac{x-s}{2}}\, dx = \int_{x_{m-2}}^{x_{m-1}} \frac{\mathscr{H}_{m-1}(x)}{\sin^2 \frac{x-s}{2}}\, dx + u'(s)\mathscr{I}_{n,m-1}(s). \tag{3.14}$$

Putting (3.12), (3.13) and (3.14) together yields (3.10) with

$$\mathscr{R}(s) = 4\mathscr{R}^{(1)}(s) + \mathscr{R}^{(2)}(s) + \mathscr{R}^{(3)}(s) + \mathscr{R}^{(4)}(s) + \bar{\mathscr{R}}(s),$$

where

$$\begin{aligned}
\mathscr{R}^{(1)}(s) &= \fint_{x_{m-1}}^{x_m} \frac{\mathscr{H}_m(x)}{(x-s)^2}\, dx, \\
\mathscr{R}^{(2)}(s) &= \fint_{x_{m-1}}^{x_m} \frac{\mathscr{H}_m(x)[\kappa_s(x) - 4]}{(x-s)^2}\, dx, \\
\mathscr{R}^{(3)}(s) &= h \sum_{\substack{i=1,\\ i \neq m, m+1}}^{n} \left[u'(\zeta_i) - u'(\eta_i)\right] \left[\cot \frac{x_i - s}{2} - \cot \frac{x_{i-1} - s}{2}\right], \\
\mathscr{R}^{(4)}(s) &= \int_{x_{m-2}}^{x_{m-1}} \frac{\mathscr{H}_{m-1}(x)}{\sin^2 \frac{x-s}{2}}\, dx,
\end{aligned}$$

and $\bar{\mathscr{R}}$ is defined in (3.1). Now we estimate $\mathscr{R}(s)$ term by term. If $u(x) \in C^{2+\alpha}[0, 2\pi]$, $(0 < \alpha \le 1)$, we get

$$|\mathscr{H}_k^{(i)}(x)| \le Ch^{2-i+\alpha}, \quad i = 0, 1, 2;\ k = m-1, m.$$

Then by using the identity,

$$\begin{aligned}
\fint_a^b \frac{u(x)}{(x-s)^2}\, dx &= \frac{(b-a)u(s)}{(b-s)(s-a)} + u'(s) \ln \frac{b-s}{s-a} \\
&\quad + \int_a^b \frac{u(x) - u(s) - u'(s)(x-s)}{(x-s)^2}\, dx
\end{aligned}$$

we have

$$\begin{aligned}\left|\mathscr{R}^{(1)}(s)\right| &\le \left|\frac{h\mathscr{H}_m(s)}{(x_m-s)(s-x_{m-1})}\right| + \left|\mathscr{H}_m'(s)\ln\frac{x_m-s}{s-x_{m-1}}\right| \\ &\quad + \left|\int_{x_{m-1}}^{x_m}\frac{1}{2}\mathscr{H}_m''(\sigma(x))\,dx\right| \\ &\le C\gamma^{-1}(h,s)h^{1+\alpha}\end{aligned}$$

where $\sigma(x)\in(x_{m-1},x_m)$. As for the second term,

$$\begin{aligned}\left|\mathscr{R}^{(2)}(s)\right| &\le \max_{x\in[x_{m-1},x_m]}|\mathscr{H}_m(x)|\int_{x_{m-1}}^{x_m}\frac{\kappa_s(x)-4}{(x-s)^2}\,dx \\ &= \max_{x\in[x_{m-1},x_m]}|\mathscr{H}_m(x)|\left\{\ne\!\!\!\!\int_{x_{m-1}}^{x_m}\frac{1}{\sin^2\frac{x-s}{2}}\,dx - \ne\!\!\!\!\int_{x_{m-1}}^{x_m}\frac{4}{(x-s)^2}\,dx\right\} \\ &= \max_{x\in[x_{m-1},x_m]}|\mathscr{H}_m(x)|\left\{-2\cot\frac{s-x_{m-1}}{2} - 2\cot\frac{x_m-s}{2}\right. \\ &\quad\left. + \frac{4h}{(x_m-s)(s-x_{m-1})}\right\} \\ &\le C\gamma^{-1}(h,s)h^{1+\alpha}.\end{aligned}$$

By noting $u(x)\in C^{2+\alpha}[0,2\pi]$, $(0<\alpha\le 1)$, one can easily get

$$\left|\mathscr{R}^{(3)}(s)\right| \le Ch^{2+\alpha}.$$

For the fourth term, we have

$$\begin{aligned}|\mathscr{R}^{(4)}(s)| &\le C\max_{0\le x\le 2\pi}\{\kappa_s(x)\}h^{1+\alpha}\int_{x_{m-2}}^{x_{m-1}}\frac{1}{(x-s)^2}\,dx \\ &\le C\rho(s)\gamma^{-1}(h,s)h^{1+\alpha}.\end{aligned}$$

The last term $\bar{\mathscr{R}}(s)$ can be estimated directly by Lemma 3.1. Putting above estimates together leads to the first two bounds in (3.11). The third bound in (3.11) can be obtained in a similar way by only noting the fact

$$|\mathscr{H}_k^{(i)}(x)| \le Ch^{3-i+\alpha}, \quad i=0,1,2;\, k=m-1,m,$$

for $u(x)\in C^{3+\alpha}[0,2\pi]$, $(0<\alpha<1)$. The proof is completed. □

The following corollary describes the superconvergence result of the composite midpoint rule $\mathscr{Q}_{0n}(s,u)$, which is a natural consequence of Theorem 3.1 by only taking into account $\tau=0$.

Corollary 3.1 *Let the composite midpoint rule $\mathcal{Q}_{0n}(s,u)$ be computed by* (2.1) *and* (2.2) *with a uniform mesh, then at $s=\hat{x}_m$, $(1 \le m \le n)$ we have*

$$|I(s,u)-\mathcal{Q}_{0n}(s,u)| \le \begin{cases} Ch^{1+\alpha}, & u(x) \in C^{2+\alpha}[0,2\pi], \\ Ch^2|\ln h|, & u(x) \in C^3[0,2\pi], \\ Ch^2, & u(x) \in C^{3+\alpha}[0,2\pi], \end{cases}$$

where $0<\alpha<1$.

At last, we suggest the modified composite midpoint rule, denoting by $\hat{\mathcal{Q}}_{0n}(s,u)$, defined by,

$$\hat{\mathcal{Q}}_{0n}(s,u) = \mathcal{Q}_{0n}(s,u) - 4\pi u'(s)\tan\frac{\tau\pi}{2}. \tag{3.15}$$

We conclude this section by the following corollary, whose proof is also a natural consequence of Theorem 3.1.

Corollary 3.2 *Let $\hat{\mathcal{Q}}_{0n}(s,u)$ be defined in* (3.15) *with a uniform mesh. Then it holds*

$$|I(s,u)-\hat{\mathcal{Q}}_{0n}(s,u)| \le \begin{cases} C\rho(s)\gamma^{-1}(h,s)h^{1+\alpha}, & u(x) \in C^{2+\alpha}[0,2\pi], \\ C\rho(s)\gamma^{-1}(h,s)h^2|\ln h|, & u(x) \in C^3[0,2\pi], \\ C\rho(s)\gamma^{-1}(h,s)h^2, & u(x) \in C^{3+\alpha}[0,2\pi] \end{cases}$$

where $0<\alpha<1$ *and* $\rho(s)$ *is defined in* (3.3).

4 A collocation scheme for hypersingular integral equation of first kind on a circle

In this section, we consider the integral equation

$$\frac{1}{4\pi} \fint_0^{2\pi} \frac{u(x)}{\sin^2\frac{x-s}{2}}\,dx = f(s), \quad s \in (0,2\pi), \tag{4.1}$$

with the compatibility condition

$$\int_0^{2\pi} f(x)\,dx = 0. \tag{4.2}$$

As in [35], we see that under the condition (4.2), there exists a unique solution up to an additive constant for the integral equation (4.1). In order to arrive at a unique solution, we adopt a periodical condition

$$\int_0^{2\pi} u(x)\,dx = 0. \tag{4.3}$$

Applying the composite midpoint rule $\mathscr{Q}_{0n}(s,u)$ to approximate the hypersingular integral in (4.1) and by collocating the resulting equation at the middle points $\hat{x}_k = x_{k-1} + h/2 (k = 1, 2, \ldots, n)$ of each subintervals, we get the following linear system

$$\frac{1}{2\pi}\sum_{m=1}^{n}\left(\cot\frac{\hat{x}_k - x_m}{2} - \cot\frac{\hat{x}_k - x_{m-1}}{2}\right)u_m = f(\hat{x}_k), \quad k = 1, 2, \ldots, n, \tag{4.4}$$

denoted by

$$\mathscr{A}_n \mathbf{U}_n^a = \mathbf{F}_n^e, \tag{4.5}$$

where

$$\begin{aligned}
\mathscr{A}_n &= (a_{km})_{n\times n},\\
a_{km} &= \frac{1}{2\pi}\left(\cot\frac{\hat{x}_k - x_m}{2} - \cot\frac{\hat{x}_k - x_{m-1}}{2}\right), \quad k, m = 1, 2, \ldots, n,\\
\mathbf{U}_n^a &= (u_1, u_2, \ldots, u_n)^T, \qquad \mathbf{F}_n^e = (f(\hat{x}_1), f(\hat{x}_2), \ldots, f(\hat{x}_n))^T,
\end{aligned}$$

and $u_k (k = 1, 2, \ldots, n)$ denote the numerical solution of u at $\hat{x}_k$. Obviously, $\mathscr{A}_n$ is a symmetric Toeplitz matrix and also a circulant matrix. However, since for any $k = 1, 2, \ldots, n$,

$$\sum_{m=1}^{n} a_{km} = \frac{1}{2\pi}\sum_{m=1}^{n}\left(\cot\frac{\hat{x}_k - x_m}{2} - \cot\frac{\hat{x}_k - x_{m-1}}{2}\right) = 0,$$

we see that $\mathscr{A}_n$ is singular, and thus the system (4.4) or (4.5) cannot be applied for solving the integral equation (4.1).

To get a well-conditioned definite system, we introduce a regularizing factor γ_{0n} in (4.4), which leads to

$$\begin{cases} \gamma_{0n} + \frac{1}{2\pi}\sum_{m=1}^{n}(\cot\frac{\hat{x}_k - x_m}{2} - \cot\frac{\hat{x}_k - x_{m-1}}{2})u_m = f(\hat{x}_k), \quad k = 1, 2, \ldots, n, \\ \sum_{m=1}^{n} u_m = 0, \end{cases} \tag{4.6}$$

where γ_{0n} have the form

$$\gamma_{0n} = \frac{1}{2\pi}\sum_{k=1}^{n} f(\hat{x}_k)h.$$

For simplicity, we describe the system (4.6) as the matrix form

$$\mathscr{A}_{n+1}\mathbf{U}_{n+1}^a = \mathbf{F}_{n+1}^e, \tag{4.7}$$

where

$$\mathscr{A}_{n+1} = \begin{pmatrix} 0 & e_n^T \\ e_n & \mathscr{A}_n \end{pmatrix},$$

$$\mathbf{U}_{n+1}^a = \begin{pmatrix} \gamma_{0n} \\ \mathbf{U}_n^a \end{pmatrix}, \qquad \mathbf{F}_{n+1}^e = \begin{pmatrix} 0 \\ \mathbf{F}_n^e \end{pmatrix},$$

and $e_n = (\underbrace{1, 1, \dots, 1}_{n})$. We rewrite the linear system (4.6) by

$$\begin{cases} \gamma_{0n} + \frac{1}{2\pi}\sum_{m=1}^{n} -\frac{u_{m+1}-u_m}{h}\cot\frac{\hat{x}_k - x_m}{2}h = f(\hat{x}_k), \quad k = 1, 2, \dots, n, \\ -\frac{1}{2\pi}\sum_{m=1}^{n}\frac{u_{m+1}-u_m}{h}h = 0, \end{cases}$$

where $u_1 = u_{n+1}$ has been used. Let $v_m = -(u_{m+1} - u_m)/h$, we get

$$\begin{cases} \gamma_{0n} + \frac{1}{2\pi}\sum_{m=1}^{n}\cot\frac{\hat{x}_k - x_m}{2}v_m h = f(\hat{x}_k), \quad k = 1, 2, \dots, n, \\ \frac{1}{2\pi}\sum_{m=1}^{n} v_m h = 0. \end{cases} \tag{4.8}$$

The following lemma has been proved in [1], which will be very important in the proof of the main result in this section.

Lemma 4.1 (Theorem 6.2.1, Sect. 6.2, Chap. 6 [1]) *For the linear system* (4.8), *its solution has the following expression*

$$v_m = -\frac{h}{2\pi}\sum_{k=1}^{n}\cot\frac{x_m - \hat{x}_k}{2}f(\hat{x}_k), \quad m = 1, 2, \dots, n. \tag{4.9}$$

Lemma 4.2 *Let $\mathscr{B}_{n+1} = (b_{ik})_{(n+1)\times(n+1)}$ be the inverse matrix of $\mathscr{A}_{n+1}$, defined in* (4.7). *Then,*

(1) *$\mathscr{B}_{n+1}$ has an explicit expression of the form*

$$\mathscr{B}_{n+1} = \begin{pmatrix} b_{00} & \boldsymbol{B}_1 \\ \boldsymbol{B}_2 & \mathscr{B}_n \end{pmatrix},$$

where

$$\boldsymbol{B}_1 = (b_{01}, b_{02}, \dots, b_{0n}), \qquad \boldsymbol{B}_2 = (b_{10}, b_{20}, \dots, b_{n0})^T,$$

$$b_{i0} = b_{0i} = \frac{1}{n}, \quad 1 \le i \le n, \tag{4.10}$$

$$b_{ik} = \frac{h^2}{2\pi}\left[\sum_{m=i}^{n-1}\cot\frac{\hat{x}_k - x_m}{2} - \frac{1}{n}\sum_{m=1}^{n-1} m\cot\frac{\hat{x}_k - x_m}{2}\right],$$

$$1 \le i \le n-1, 1 \le k \le n, \tag{4.11}$$

$$b_{nk} = -\frac{h^2}{2n\pi}\sum_{m=1}^{n-1} m\cot\frac{\hat{x}_k - x_m}{2}, \quad 1 \le k \le n. \tag{4.12}$$

(2) *$\mathscr{B}_n$ is a Toeplitz matrix, and also a circulant matrix.*
(3) *For $i = 1, 2, \dots, n$, there exist a positive constant C such that*

$$\sum_{k=1}^{n}|b_{ik}| \le C. \tag{4.13}$$

Proof (1) From the last equation in (4.6), we see that

$$0=\sum_{m=1}^{n}u_m=-h\sum_{m=1}^{n-1}m\frac{u_{m+1}-u_m}{h}+nu_n=h\sum_{m=1}^{n-1}mv_m+nu_n,$$

by using (4.9), we have

$$u_n=-\frac{h}{n}\sum_{m=1}^{n-1}mv_m=-\frac{h^2}{2n\pi}\sum_{m=1}^{n-1}\sum_{k=1}^{n}m\cot\frac{\hat{x}_k-x_m}{2}f(\hat{x}_k),$$

which leads to (4.12). Then, also by using (4.9), we get

$$\begin{aligned}u_i&=h\sum_{m=i}^{n-1}v_m+u_n\\&=\frac{h^2}{2\pi}\sum_{m=i}^{n-1}\sum_{k=1}^{n}\cot\frac{\hat{x}_k-x_m}{2}f(\hat{x}_k)-\frac{h^2}{2n\pi}\sum_{m=1}^{n-1}\sum_{k=1}^{n}m\cot\frac{\hat{x}_k-x_m}{2}f(\hat{x}_k)\\&=\frac{h^2}{2\pi}\sum_{k=1}^{n}\left[\sum_{m=i}^{n-1}\cot\frac{\hat{x}_k-x_m}{2}-\frac{1}{n}\sum_{m=1}^{n-1}m\cot\frac{\hat{x}_k-x_m}{2}\right]f(\hat{x}_k),\end{aligned}$$

which yields (4.11). The proof of (4.10) will be given later.

(2) Since

$$\begin{aligned}-\frac{1}{n}\sum_{m=1}^{n-1}m\cot\frac{\hat{x}_{k+1}-x_m}{2}&=-\frac{1}{n}\sum_{m=1}^{n-1}m\cot\frac{\hat{x}_k-x_{m-1}}{2}\\&=-\frac{1}{n}\sum_{m=0}^{n-2}(m+1)\cot\frac{\hat{x}_k-x_m}{2}\\&=-\frac{1}{n}\left[\sum_{m=1}^{n-2}m\cot\frac{\hat{x}_k-x_m}{2}+\sum_{m=0}^{n-2}\cot\frac{\hat{x}_k-x_m}{2}\right]\\&=-\frac{1}{n}\left[\sum_{m=1}^{n-1}m\cot\frac{\hat{x}_k-x_m}{2}+\sum_{m=0}^{n-1}\cot\frac{\hat{x}_k-x_m}{2}\right.\\&\qquad\left.-n\cot\frac{\hat{x}_k-x_{n-1}}{2}\right]\\&=\cot\frac{\hat{x}_k-x_{n-1}}{2}-\frac{1}{n}\sum_{m=1}^{n-1}m\cot\frac{\hat{x}_k-x_m}{2},\end{aligned}\tag{4.14}$$

therefore, for $i = 1, 2, \ldots, n-2$,

$$b_{i+1,k+1} = \frac{h^2}{2\pi}\left[\sum_{m=i+1}^{n-1} \cot\frac{\hat{x}_{k+1} - x_m}{2} - \frac{1}{n}\sum_{m=1}^{n-1} m \cot\frac{\hat{x}_{k+1} - x_m}{2}\right]$$

$$= \frac{h^2}{2\pi}\left[\sum_{m=i}^{n-2} \cot\frac{\hat{x}_k - x_m}{2} + \cot\frac{\hat{x}_k - x_{n-1}}{2} - \frac{1}{n}\sum_{m=1}^{n-1} m \cot\frac{\hat{x}_k - x_m}{2}\right]$$

$$= b_{ik}, \tag{4.15}$$

where

$$\sum_{m=0}^{n-1} \cot\frac{\hat{x}_k - x_m}{2} = 0$$

has been used. Moreover, (4.14) implies

$$b_{n,k+1} = b_{n-1,k}, \qquad \text{for } k = 1, 2, \ldots, n-1. \tag{4.16}$$

Combining (4.15) and (4.16) show that $\mathscr{B}_n$ is a Toeplitz matrix. Moreover, since

$$b_{1k} = \frac{h^2}{2\pi}\left[\sum_{m=1}^{n-1} \cot\frac{\hat{x}_k - x_m}{2} - \frac{1}{n}\sum_{m=1}^{n-1} m \cot\frac{\hat{x}_k - x_m}{2}\right]$$

$$= \frac{h^2}{2\pi}\left[\sum_{m=1}^{n} \cot\frac{\hat{x}_k - x_m}{2} - \frac{1}{n}\sum_{m=1}^{n} m \cot\frac{\hat{x}_k - x_m}{2}\right]$$

$$= -\frac{h^2}{2n\pi}\sum_{m=1}^{n} m \cot\frac{\hat{x}_k - x_m}{2},$$

and

$$b_{n,k-1} = -\frac{h^2}{2n\pi}\sum_{m=1}^{n-1} \cot\frac{\hat{x}_{k-1} - x_m}{2} = -\frac{h^2}{2n\pi}\sum_{m=1}^{n-1} \cot\frac{\hat{x}_k - x_{m+1}}{2}$$

$$= -\frac{h^2}{2n\pi}\sum_{m=1}^{n-1} (m+1) \cot\frac{\hat{x}_k - x_{m+1}}{2} + \frac{h^2}{2n\pi}\sum_{m=1}^{n-1} \cot\frac{\hat{x}_k - x_{m+1}}{2}$$

$$= -\frac{h^2}{2n\pi}\sum_{m=2}^{n} m \cot\frac{\hat{x}_k - x_m}{2} + \frac{h^2}{2n\pi}\sum_{m=2}^{n} \cot\frac{\hat{x}_k - x_m}{2}$$

$$= -\frac{h^2}{2n\pi}\sum_{m=1}^{n} m \cot\frac{\hat{x}_k - x_m}{2} + \frac{h^2}{2n\pi}\sum_{m=1}^{n} \cot\frac{\hat{x}_k - x_m}{2}$$

$$= -\frac{h^2}{2n\pi}\sum_{m=1}^{n} m \cot\frac{\hat{x}_k - x_m}{2},$$

we have $b_{n,k-1} = b_{1k}$ for $k = 2, 3, \dots, n$, which show that $\mathscr{B}_n$ is a circulant matrix by noting that $\mathscr{B}_n$ is a Toeplitz matrix.

Now we prove (4.10). Since $\mathscr{B}_{n+1}$ is the inverse matrix of $\mathscr{A}_{n+1}$, and $\mathscr{A}_{n+1}$ is symmetric, we see that $\mathscr{B}_{n+1}$ is also symmetric, i.e., $B_1^T = B_2$, in other word, for $j = 1, \dots, n$,

$$b_{j0} = b_{0j}. \tag{4.17}$$

By multiplying the ith row of $\mathscr{B}_{n+1}$ with the ith column of $\mathscr{A}_{n+1}$ and noting the circulant property of their submatrix $\mathscr{A}_n$ and $\mathscr{B}_n$, we obtain

$$b_{i0} + \sum_{j=1}^{n} b_{ij} a_{ji} = 1, \quad 1 \le i \le n,$$

i.e.,

$$b_{i0} = 1 - \sum_{j=1}^{n} b_{ij} a_{ji}, \quad 1 \le i \le n, \tag{4.18}$$

are equal. By multiplying the first row of $\mathscr{B}_{n+1}$ with the first column of $\mathscr{A}_{n+1}$, we see that

$$\sum_{j=1}^{n} b_{0j} = 1. \tag{4.19}$$

Combining (4.17), (4.18) and (4.19) leads to $b_{i0} = b_{0k} = 1/n$.

(3) Since $\mathscr{B}_n$ is a circulant matrix, to prove (4.13), we just need to consider the case $k = n$. By straightforward computation, we have

$$b_{nk} = \frac{1}{2n\pi} \sum_{m=1}^{n} (\hat{x}_m - x_{k-1}) \cot \frac{\hat{x}_m - x_{k-1}}{2} h + \frac{h}{n} \cot \frac{x_{k-1} - \hat{x}_n}{2}. \tag{4.20}$$

We can see that the first term in the right-hand of (4.20) is just the midpoint quadrature of the integral

$$\frac{1}{2n\pi} \int_0^{2\pi} (x - s) \cot \frac{x - s}{2}\, dx = \frac{1}{2n\pi} \left[J_1(2\pi - s) + J_1(s) \right], \tag{4.21}$$

with $s = x_{k-1}$, where the identity (see [3])

$$J_1(s) = \int_0^s t \cot \frac{t}{2}\, dt = 2s \ln \left(2 \sin \frac{s}{2} \right) + 2\mathrm{Cl}_2(s), \tag{4.22}$$

has been used, and

$$\mathrm{Cl}_2(s) = -\int_0^s \ln \left(2 \sin \frac{t}{2} \right) dt = \sum_{k=1}^{\infty} \frac{\sin(ks)}{k^2}$$

is the Clausen's integral [33]. Since the integrand function in (4.21) is continuous except one removable discontinuity at s, by the error estimate of the midpoint for Riemann integrals, we have

$$\frac{1}{2n\pi}\sum_{m=1}^{n}(\hat{x}_m - x_{k-1})\cot\frac{\hat{x}_m - x_{k-1}}{2h} = \frac{1}{2n\pi}[J_1(2\pi - x_{k-1}) + J_1(x_{k-1})] + O(h^3)$$

By (4.22), we see that $J_1(s)$ is continuous, and thus for any $k = 1, 2, \ldots, n$,

$$|b_{nk}| \le \frac{1}{2n\pi}\left[|J_1(2\pi - x_{k-1})| + |J_1(x_{k-1})|\right] + O(h^3) + \frac{h}{n}\cot\frac{h}{4} \le \frac{C}{n}, \tag{4.23}$$

where the inequality

$$\cot\frac{\hat{x}_n - x_{k-1}}{2} \le \cot\frac{h}{4}, \quad k = 1, 2, \ldots, n$$

has been used. Therefore, (4.13) can be obtained directly from (4.23). □

Now we present our main result of this section.

Theorem 4.1 *Assume that $u(x)$, the solution of the hypersingular integral equation (4.1), belongs to $C^{3+\alpha}[0, 2\pi]$, $(0 < \alpha < 1)$. Then, for the solution of linear system (4.6) or (4.7), there holds the error estimate*

$$\max_{1\le i\le n}|u(\hat{x}_i) - u_i| \le Ch^2.$$

Proof Let $\mathbf{U}^e_{n+1} = (0, u(\hat{x}_1), u(\hat{x}_2), \ldots, u(\hat{x}_n))^T$ be the exact vector. Then, from (4.7), we get

$$\mathbf{U}^e_{n+1} - \mathbf{U}^a_{n+1} = \mathscr{B}_{n+1}(\mathscr{A}_{n+1}\mathbf{U}^e_{n+1} - \mathbf{F}^e_{n+1}),$$

which implies

$$\begin{aligned} u(\hat{x}_i) - u_i &= b_{i0}\sum_{m=1}^{n}u(\hat{x}_m) + \sum_{k=1}^{n}b_{ik}\left[\sum_{m=1}^{n}a_{km}u(\hat{x}_m) - f(\hat{x}_k)\right] \\ &= b_{i0}\sum_{m=1}^{n}u(\hat{x}_m) + \sum_{k=1}^{n}b_{ik}\mathscr{E}_{0n}(\hat{x}_k, u), \quad i = 1, 2, \ldots, n, \end{aligned}$$

where $\{b_{ik}\}$ are the entries of $\mathscr{B}_{n+1}$ and $\mathscr{E}_{0n}(\hat{x}_k, u)$ is defined in (2.1). Here, we have used the fact that $\sum_{m=1}^{n}a_{km}u(\hat{x}_m)$ is just the midpoint rule of the hypersingular integral in the left-hand of (4.1). By Corollary 3.1, (4.10) and (4.13), we obtain

$$|u(\hat{x}_i) - u_i| \le \frac{1}{2\pi}\left|\sum_{m=1}^{n}u(\hat{x}_m)h\right| + \sum_{k=1}^{n}|b_{ik}||\mathscr{E}_{0n}(\hat{x}_k, u)|$$

Table 1 Errors of the case where $s = x_{[n/4]} + (1+\tau)h/2$

n	$\tau = -1/3$	$\tau = 0$	$\tau = 1/3$	$\tau = -1/3$	$\tau = 0$	$\tau = 1/3$
32	0.12856E-01	0.36204E-01	0.60881E-01	0.15441E+02	0.36204E-01	0.16219E+02
64	0.37422E-02	0.95846E-02	0.15606E-01	0.14981E+02	0.95846E-02	0.15413E+02
128	0.10012E-02	0.24605E-02	0.39430E-02	0.14747E+02	0.24605E-02	0.14973E+02
256	0.25847E-03	0.62303E-03	0.99053E-03	0.14629E+02	0.62303E-03	0.14745E+02
512	0.65636E-04	0.15674E-03	0.24820E-03	0.14570E+02	0.15674E-03	0.14628E+02
h^α	1.903	1.963	1.985	–	1.963	–

$$\leq Ch^2 + Ch^2 \sum_{k=1}^{n} |b_{ik}| \leq Ch^2,$$

where the fact $\sum_{m=1}^{n} u(\hat{x}_m)h$ is the midpoint rule of the Riemann integral (4.3) with accuracy $O(h^2)$ has been used. The proof is completed. □

5 Numerical Experiment

In this section, we present some numerical examples to confirm our theoretical results given in the above sections.

Example 5.1 Consider hypersingular integral (1.1) with a smooth density function $u(x) = 2\cos x + 2\sin x$ and $c = 0$. The exact value of this finite-part integral is $-8\pi(\cos s + \sin s)$. Here we use the midpoint rule (2.1) and the modified midpoint rule (3.15) to compute this integral. Numerical results are presented in Table 1 for a dynamic singular point $s = x_{[n/4]} + (1+\tau)h/2$ and Table 2 for $s = x_{n-1} + (1+\tau)h/2$. The estimated orders of convergence are given in the last row, which are obtained by a least squares fit. The left half of Table 1 shows that the accuracy of the modified midpoint rule is always $O(h^2)$, independent of the values of τ. However, from the right half, we see that if s is not a superconvergence point ($\tau \neq 0$), the midpoint rule is divergent, while the accuracy is $O(h^2)$ if s is located at the superconvergence point ($\tau = 0$). Table 2 describes the case when s approaches the endpoints as h goes to zero, which have the same result as Table 1. These results agree quite well with the theoretical results in Corollary 3.1 and Corollary 3.2.

Example 5.2 Here we consider an example with less regularity. Let

$$I(s,u) = ⨍_{-\pi}^{\pi} \frac{u(x)}{\sin^2 \frac{x-s}{2}}\,dx, \quad s \in (-\pi, \pi),$$

where $s = 0$ and

$$u(x) = |x(x^2 - \pi^2)|^{p+\alpha}, \quad p = 1, 2, 3,$$

and we extend it to be a periodic function, still denoted by $u(x)$, with period 2π by taking $u(x) = u(x + 2\pi)$. Obviously, $u(x) \in C^{p+\alpha}[-\pi, \pi]$. Since the exact value cannot obtained by an analytical process, we use the trapezoidal rule discussed in

Table 2 Errors of the case where $s = x_{n-1} + (1+\tau)h/2$

n	$\hat{\mathscr{Q}}_{0n}(s,u)$			$\mathscr{Q}_{0n}(s,u)$		
	$\tau = -1/3$	$\tau = 0$	$\tau = 1/3$	$\tau = -1/3$	$\tau = 0$	$\tau = 1/3$
32	0.60881E-01	0.36204E-01	0.12856E-01	0.16219E+02	0.36204E-01	0.15441E+02
64	0.15606E-01	0.95846E-02	0.37422E-02	0.15413E+02	0.95846E-02	0.14981E+02
128	0.39430E-02	0.24605E-02	0.10012E-02	0.14973E+02	0.24605E-02	0.14747E+02
256	0.99053E-03	0.62303E-03	0.25847E-03	0.14745E+02	0.62303E-03	0.14629E+02
512	0.24820E-03	0.15674E-03	0.65636E-04	0.14628E+02	0.15674E-03	0.14570E+02
h^α	1.985	1.963	1.903	–	1.963	–

Table 3 Errors of the case where $s = x_{n-1} + (1+\tau)h/2$

n	$p = 1$	$p = 2$	$p = 3$
31	0.11688E+03	0.74480E+02	0.43098E+03
63	0.81281E+02	0.30441E+02	0.11406E+03
127	0.56984E+02	0.11774E+02	0.29674E+02
255	0.40065E+02	0.44184E+01	0.76389E+01
511	0.28182E+02	0.16280E+01	0.19534E+01
h^α	0.508	1.421	1.926

[38] by using a mesh with size small enough to get the integral value. We can see from Table 3 that if the density function $u(x)$ is smooth enough ($p = 3$), the error bound is $O(h^2)$, and if $u(x)$ has less regularity ($p = 1, 2$), the error bound is about $O(h^{p-1+\alpha})$, which confirms our estimate given in Corollary 3.1 is optimal.

Example 5.3 Now we consider an example of solving hypersingular integral equation (4.1) by collocation scheme (4.6). Let $c = 0$ and $f(s) = -2(\cos 2s + \sin 2s)$. The exact solution is $u(x) = \cos 2x + \sin 2x$. We examine the maximal nodal error and the maximal truncation error, defined by

$$e_\infty = \max_{1\le i\le n} |u(\hat{x}_i) - u_i|, \text{trunc} - e_\infty = \max_{1\le i\le n} |\mathscr{E}_{0n}(\hat{x}_i, u)|,$$

respectively, where u_i denotes the approximation of $u(x)$ at $\hat{x}_i$ and $\mathscr{E}_{0n}(\hat{x}_i, u)$ is defined in (2.1). Numerical results presented in Table 4 show that both the maximal nodal error and the maximal truncation error are $O(h^2)$, which is in good agreement with the result in Theorem 4.1.

Example 5.4 At last, we give an example to compare the current method with the indirect method based on trapezoidal rule discussed in [38] for solving (4.1). For convenience and completeness, we describe the indirect method as follows

$$\frac{1}{4\pi}\sum_{j=1}^{n} \frac{1}{s_{i2} - s_{i1}} \left[\omega_j^1(s_{i2})(x_i - s_{i1}) + \omega_j^1(s_{i1})(s_{i2} - x_i)\right] u_j = f(x_i),$$
$$i = 1, 2, \ldots, n, \tag{5.1}$$

Table 4 L_∞ errors of the midpoint rule for solving (4.1)

n	e_∞	$\text{trunc} - e_\infty$
32	0.89527E-02	0.22356E+00
64	0.22634E-02	0.56793E-01
128	0.56742E-03	0.14255E-01
256	0.14195E-03	0.35673E-02
512	0.35494E-04	0.89205E-03
h^α	1.993	1.995

Table 5 L_∞ errors of the indirect method (5.1) for solving (4.1)

n	S_1	S_2
32	0.203586E-01	0.894043E-01
64	0.518539E-02	0.375929E-01
128	0.131010E-02	0.170495E-01
256	0.329369E-03	0.809080E-02
512	0.825804E-04	0.393718E-02
h^α	1.987	1.126

where

$$\omega_i^1(s) = \frac{4}{h} \ln \left| \frac{1 - \cos(x_i - s)}{\cos h - \cos(x_i - s)} \right|$$

is Cotes coefficients of the trapezoidal rule. The linear system (5.1) is ill-conditioned, and we take the same regularization process as in (4.6) to get a well-conditioned definite system. Moreover, we adopt a uniform mesh and get the linear system with the collocation points as nodal ones and the set of s_{i1} and s_{i2}

$$S_1 = \{s_{i1} = t_{i-1} + h/2 + h/3,\ s_{i2} = t_i + h/2 - h/3\},$$

$$S_2 = \{s_{i1} = t_{i-1} + h/2 + h/4,\ s_{i2} = t_i + h/2 - h/4\}.$$

Here, S_1 consists of the superconvergence points, and S_2 contains non-superconvergence ones. From Tables 4 and 5, we see that the accuracy of the midpoint rule and the indirect method (5.1) for solving (4.1) are both $O(h^2)$ if their corresponding superconvergence points are chosen, but errors of the midpoint rule seem smaller. In fact, compared with the indirect method (5.1), the midpoint rule is more competitive due to its flexibility and simplicity.

6 Conclusion

We have shown both theoretically and numerically the superconvergence of the midpoint rule for the evaluation of hypersingular integrals with the kernel $\sin^{-2}[(x-s)/2]$ defined on a circle. For general values of the singular point s, the rule is divergent, but if s is located at the midpoint of each subinterval, its superconvergence phenomenon occurs. Moreover, we have applied the rule and its superconvergence result for solving the corresponding hypersingular integral equation of the first kind via collocation method by choosing midpoints as collocation points. However, the linear system generated by this means is singular. By introducing a regularizing factor, a well-conditioned definite system is obtained. More importantly, the inverse of coefficient matrix for this new system possesses an explicit expression, by which an optimal error estimate is established.

In [32], the authors firstly studied the superconvergence of the composite midpoint rule for the evaluation of hypersingular integrals defined on an interval with kernel $(x-s)^{-2}$, and then applied these results to numerical solution of corresponding hypersingular integral equation. The coefficient matrix of its resulting system has many good properties, such as symmetric Toeplitz type, M-type, persymmetric type, and its inverse also possesses an explicit expression. But it's just a first-order method, although a second-order accuracy can be obtained for the evaluation of the integrals. The reason why this phenomenon happens is due to the impact of the endpoints on the errors, since the midpoint rule is always divergent if the singular point s is very close to the endpoints, even at the midpoints. It should be pointed out that for the rule in this paper the aforementioned phenomenon disappears, since the kernel and the density function are both 2π-periodic. Thus the rule in such case is of second-order for both evaluation of hypersingular integrals and solution of hypersingular integral equations.

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